"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed.

1) T **F** All vectors $\vec{v}$ in space define a direction $\vec{v}/|\vec{v}|$.

2) **T** F The vectors $[1, 3, 1]$ and $[1, -1, 2]$ are perpendicular.

3) **T** F The curvature of the curve $\vec{r}(t) = [\sin(\sin(t)), \cos(\sin(t)), 3]$ is 1 at $t = 0$.

4) **T** F Clairaut's theorem assures that $f_{xyx} = f_{yxx}$ for every smooth function $f(x, y)$.

Solution:
It must be differentiable.

5) **T** F Maxwell took an exam in which he had to prove Stokes theorem.

Solution:
We saw this story

6) **T** F The boundary of the ball $x^2 + y^2 + z^2 \leq 1$ is the sphere $x^2 + y^2 + z^2 = 1$.

Solution:
A sphere and a ball are different things

7) T **F** The boundary of the sphere $x^2 + y^2 + z^2 = 1$ is the circle $x^2 + y^2 = 1, z = 0$.

Solution:
The boundary of sphere is empty.

8) T **F** The divergence of the gradient of the divergence of a field is always zero. In other words: $\text{div}(\text{grad}(\text{div}(\vec{F}))) = 0$.

Solution:
Take $F = [x^3, 0, 0]$. Then $div(\vec{F}) = 3x^2$ its gradient is $[6x^2, 0, 0]$ and the divergence is 12x.

9) **T** | F

We can use the divergence theorem to compute the volume of a solid.

Solution:
Take $F = [x, 0, 0]$ for example, then $div(F) = 1$.

10) **T** | F

The line $\vec{r}(t) = [t, 2t, -3t]$ hits the plane $x + 2y - 3z = 10$ orthogonally.

Solution:
Yes, the normal vector to the plane is $[1, 2, -3]$.

11) T | **F**

The quadratic surface $x^2 + y^2 - z^2 = -1$ is a one-sheeted hyperboloid.

Solution:
It is a two sheeted hyperboloid.

12) **T** | F

For non-zero vectors $\vec{u}, \vec{v}$, the relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ implies that $\vec{u}, \vec{v}$ meet in a $\pi/4$ angle.

Solution:
We know this from the formula for the dot product and cross product

13) T | **F**

$\int_0^{\pi/2} \int_0^3 r^5 \, d\theta \, dr = \int_0^3 \int_0^3 (x^2 + y^2)^2 \, dxdy$.

Solution:
The bounds are wrong. The second integral integrates over a square.

14) T | **F**

If $\vec{F} = [P, Q]$ satisfies $\text{curl}(\vec{F})(x, y) = 0$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all points (x, y), then $\vec{F}$ is a constant field.

Solution:
Also other fields like $\vec{F}(x, y) = [x + y, x - y]$ satisfy this.

15) | T | F (highlighted) |

The acceleration $\vec{r}\,''(t) = [x(t), y(t), z(t)]$, the velocity vector $\vec{r}\,'(t)$ and $\vec{r}\,'(t) \times \vec{r}\,''(t)$ are pairwise perpendicular.

Solution:
Already for a straight line, this is not the case.

16) | T (highlighted) | F |

The curvature of the curve $\vec{r}(t) = [\sin(2t), 0, \cos(2t)]$ is equal to the curvature of the curve $\vec{s}(t) = [0, \cos(3t), \sin(3t)]$.

Solution:
Both are circles of radius 1.

17) | T (highlighted) | F |

The space curve $\vec{r}(t) = [t\sin(t), t\cos(t), t^2]$ for $t \in [0, 10\pi]$ is located on a paraboloid.

Solution:
Indeed $x(t)^2 + y(t)^2 = z(t)$.

18) | T (highlighted) | F |

If a smooth function $f(x, y)$ has a global maximum, then this maximum is a critical point.

Solution:
It is then also a local maximum.

19) | T (highlighted) | F |

If $L(x, y)$ is the linearization of $f(x, y)$ and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ at t_0. Then $d/dtL(\vec{s}(t)) = d/dtf(\vec{r}(t))$ at the time $t = t_0$.

Solution:
This is how the chain rule can be proved.

20) | T (highlighted) | F |

The flux of an incompressible vector field is zero through any closed surface S in space.

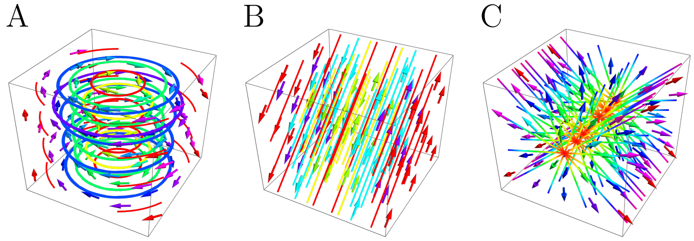
Solution:
By the divergence theorem.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) The figures display vector fields in space. There is an exact match.

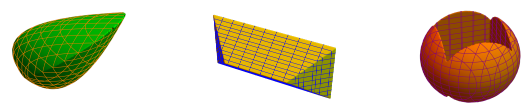
Field	A-C
$\vec{F} = [0, x, x]$	
$\vec{F} = [y, -x, 0]$	
$\vec{F} = [x, 0, z]$	

A B C



b) (2 points) Match the solids. There is an exact match.

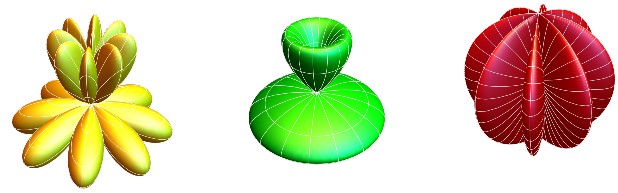
A B C



Solid	A-C
$x^2 + y^2 + z^2 < 5, \lvert x\rvert + \lvert y\rvert > 2$	
$\lvert x\rvert + \lvert y\rvert < 2, \lvert y\rvert + \lvert z\rvert < 1$	
$x^2 - y^2 < 1 + z, y^2 + z^2 < 1 + x$	

c) (2 points) Match the surfaces

A B C



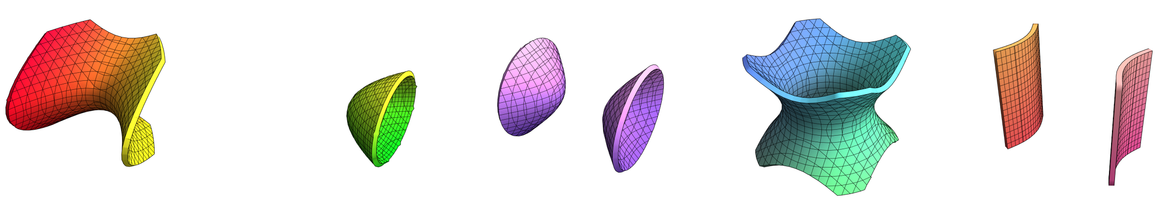
Surface	A-C
$\rho = \sin(4\phi)$	
$\rho = \sin(4\theta) + \sin(4\phi)$	
$\rho = \sin(4\theta)$	

d) (2 points) What is the name and formula of the partial differential equation is used to model waves reaching a beach?

Name:	Equation:

e) (2 points) Just to double check that you still have all your ~~marbles~~ quadrics in your head:

	Enter a letter from A-E in each of the three cases
Pick the hyperbolic paraboloid	
Pick the two-sheeted hyperboloid	
Pick the cylindrical hyperboloid	



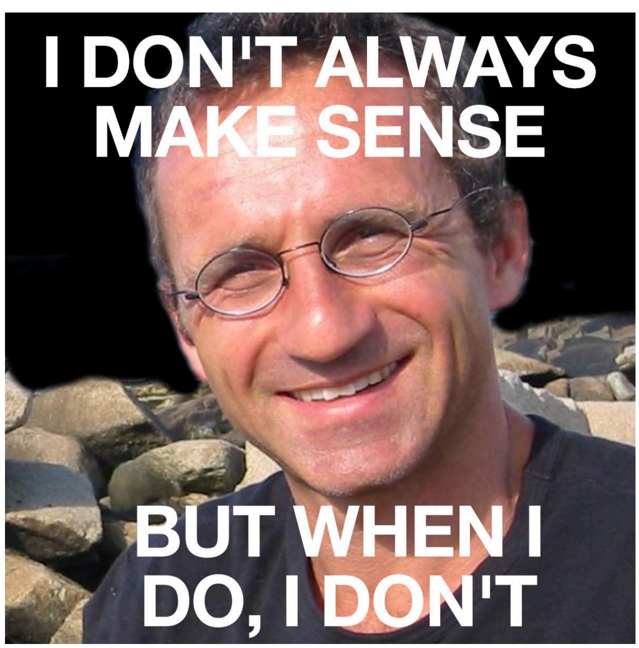
A B C D E

Solution:
a) BAC
b) CBA
c) BAC
d) Burgers $u_t + uu_x = u_{xx}$ or $u_t = uu_t$ or $u_t + uu_t = 0$.
e) ACE

Problem 3) (10 points) No justifications necessary

Which of the following expressions make sense? Each question is one point.

Expression	Makes sense	Does not make sense
The double integral $\iint x^2 \, dxdy$		
$\iiint \text{div}(\text{grad}(x^2+y^2+z^2)) \, dxdydz$		
The vector field $\text{grad}(\text{curl}(\vec{F}))$		
The distance between two lines		
The distance between two vector fields		
The directional derivative $D_{\vec{v}}\vec{F}$		
The flow line of a parametrized surface		
The traces of an implicit surface		
The flux of a scalar function f through a sphere		
The arc length of a closed flow line of a vector field		



Solution:
1,2,4,8,10 make sense. The others not. (In 3, it was accepted to have it make sense as it does make sense in 2 dimensions).

Problem 4) (10 points)

Noah Lyles recently clocked 19.31 seconds for the 200 meters sprint in Oregon. This is faster than the 19.32 of Michael Johnson in 1996. The first part of the sprint are run on a semi-circle or radius 36.50 meters. We look at a fellow runner who follows a circle of radius 40 meters, meaning

$$\vec{r}(t) = [40\cos(at), 40\sin(at), 0] \ .$$

Assuming this runner runs the 200 meters in 20 seconds, his speed is $|\vec{r}\,'(t)| = a40 = 10$ in meters per second, giving him an angular velocity of $a = 10/40 = 1/4$. The upshot of this introduction is that we have a precise curve tracing the runner:

$$\vec{r}(t) = [40\cos(\frac{t}{4}), 40\sin(\frac{t}{4}), 0] \ .$$

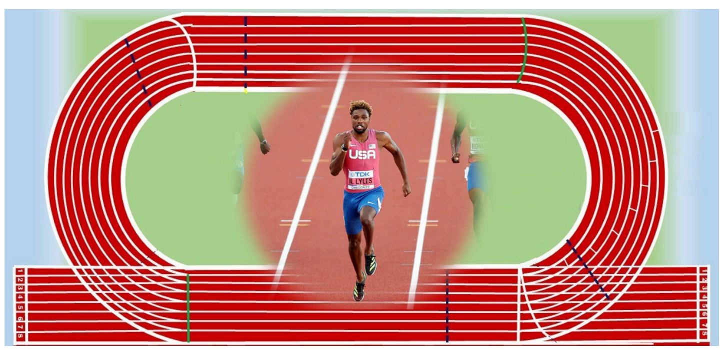
a) (2 points) What is the curvature of the path at $t = \pi$?

b) (2 points) Compute the vectors $\vec{T}, \vec{N}, \vec{B}$ at $t = \pi$.

c) (2 points) Use the arc length formula to get the arc length for $t \in [0, \pi/2]$.

d) (2 points) What is the magnitude of the acceleration $|r''(t)|$ at $t = \pi$?

e) (2 points) Finally compute the jerk $\vec{r}\,'''(t)$ at $t = \pi$.



Solution:
a) 1/40 (Note that this is a circle of radius 40. There was no need to compute the curvature from the formula, even so some did.)
b) $T = [-1, 1, 0]/\sqrt{2}$, $N = [1, 1, 0]/\sqrt{2}$ and $B = [0, 0, 1]$.
c) $|r'| = 10$ gives $\int_0^{\pi/2} 10\, dt = 5\pi$. Note that the speed had already been given.
d) 5/2 by direct computation.
e) $[1, -1, 0]5\sqrt{2}/16$ also by direct computation.

Problem 5) (10 points)

The curve

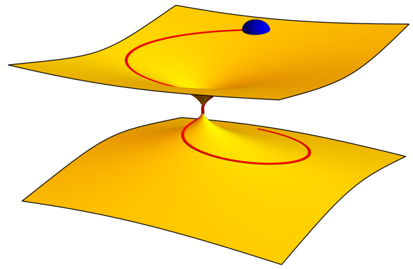
$$\vec{r}(t) = [1 + t^2\sin(\pi t), t^2\cos(\pi t), t]$$

is located on the surface S given by the implicit equation

$$(x - 1)^2 + y^2 = z^4 \ .$$

a) (5 points) Find the tangent plane $ax + by + cz = d$ to the surface at the point $P = (1, 4, 2)$.

b) (5 points) Parametrize the tangent line through the point P containing the velocity vector of $\vec{r}(t)$ at $\vec{r}(2) = (1, 4, 2)$.



Solution:
a) $f = (x-1)^2 + y^2 - z^4$. $\nabla f = [2(x-1), 2y, -5z^4]$. At the point $(1, 4, 2)$ this is $\nabla f(1, 4, 2) = [0, 8, -32]$. So $8y - 32z = 32$ (The constant 32 is obtained by plugging in the point).
b) The velocity at $t = 2$ is $\vec{v} = [4\pi, 4, 1]$. The line is $r(t) = [1, 4, 2] + t[4\pi, 4, 1]$.

Problem 6) (10 points)

a) (3 points) Find the **surface area** of the parallelogram with vertices

$$A = (0, 0, 0), B = (1, 1, 0), C = (2, 3, 3), D = (1, 2, 3)$$

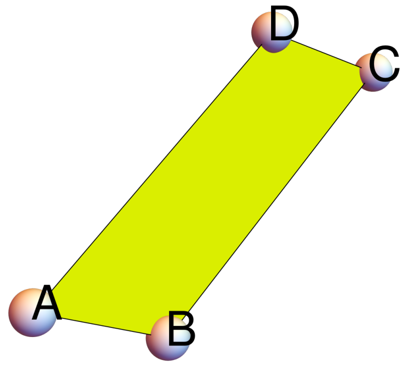
without integrating. Just compute it using a suitable cross product.

b) (3 points) Now find the surface area by using the parametrization

$$\vec{r}(t, s) = [t + s, t + 2s, 3s], 0 \leq t \leq 1, 0 \leq s \leq 1$$

and writing down an integral. We need in this part b) to see a double integral, and not just the answer.

c) (4 points) What is the distance from C to the line through A, B?

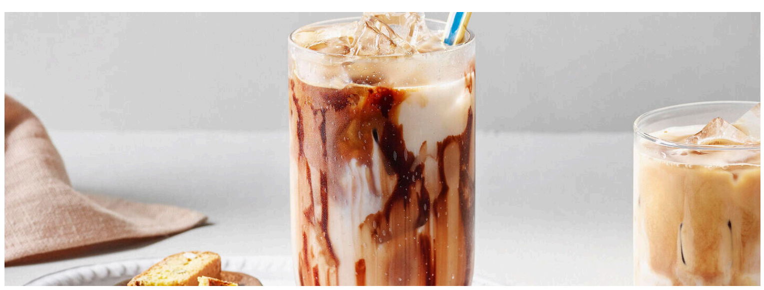


Solution:
a) $\sqrt{19}$
b) $\int_0^1 \int_0^1 \sqrt{19}\, dsdt = \sqrt{19}$.
c) Distance = Area/(base length) = $\sqrt{19}/\sqrt{2}$.

Problem 7) (10 points)

Just after the second midterm, we started to compute **triple integrals**. Let us remember this wonderful moment because the hours after an exam are the sweetest in any school. We will also enjoy the afternoon after this final. What is nicer than slurping on an **iced vanilla caramel latte**, while Oliver has to grade exams on a hot day of 37 degrees celsius. Compute the following 3D integral:

$$\int_0^{\pi/2} \int_y^{\pi/2} \int_0^x \frac{\cos(2z)}{2x}\, dz\, dx\, dy\ .$$



Solution:
Integrate out the most inner integral to get

$$\int_0^{\pi/2} \int_y^{\pi/2} \sin(2x)/x \; dxdy \; .$$

Now we are stuck. We need to change the order of integration to continue. But this is a routine situation:

$$\int_0^{\pi/2} \int_0^{x} \sin(2x)/x \; dydx \; .$$

The answer is 1/4.

Problem 8) (10 points)

In an other attempt to derail a computer algebra system we give it the problem to classify the critical points of the infamous **1001-function**. Oliver has mentioned this example already in class. He likes to collect problems in which computers **suck** and get **stuck** but humans **succeed** and **lead**. He also likes to **rhyme** from time to **time**.

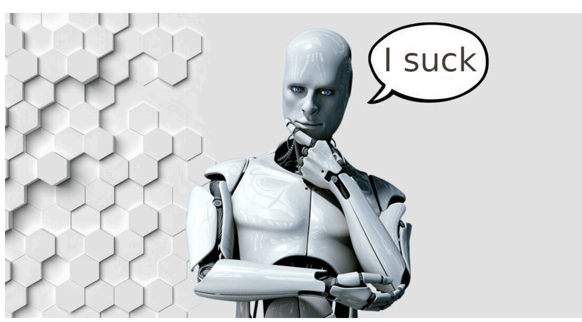
$$f(x,y) = x - \frac{x^{1001}}{1001} + y - \frac{y^{1001}}{1001} \; .$$

We also want to know whether the function f has a global max or min.

Solution:

x	y	D	f_{xx}	Nature	Value of f
-1	-1	10000	100	minimum	-(200/101)
-1	1	-10000	100	saddle	0
1	-1	-10000	-100	saddle	0
1	1	10000	-100	maximum	200/101

There is no global maximum nor global minimum as one can see already by putting $y = 0$.

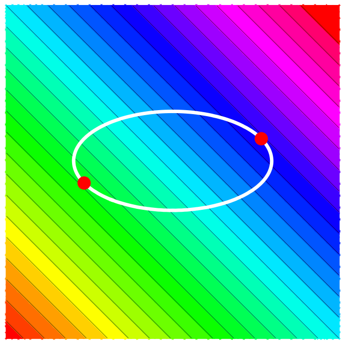


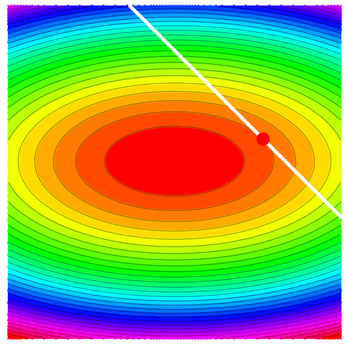
Problem 9) (10 points)

We would naively expect that extremizing f under the constraint g is equivalent to extremizing g under constraing when f is constant. This is not the case as we can see here:

a) (5 points) Find the minimum of $f(x,y) = 4 + x^2 - 8y + 4y^2$ under the constraint $g(x,y) = x + y = 11$ using Lagrange's method. You see only one solution.

b) (5 points) Now find the maximum and minimum of $f(x,y) = x + y$ under the constraint $g(x,y) = 4 + x^2 - 8y + 4y^2 = 80$ again using the method of Lagrange multipliers. You see two solutions of the Lagrange equations. Identify which is the maximum and which is the minimum.





Solution:
a) This is a routine Lagrange problem. The solution is (8,3)
b) There are two solutions (8,3) and (-8,1). The first one is the maximum, the second the minimum.

Problem 10) (10 points)

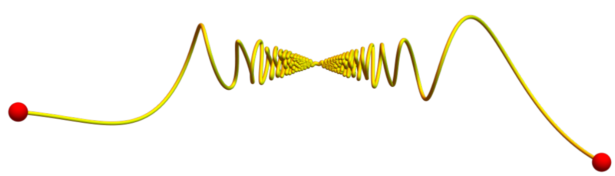
Find the **line integral** $\int_C \vec{F} \cdot d\vec{r}$ for

$$\vec{F}(x,y,z) = [x^2 + yz, y^2 + xz, z^2 + xy]$$

along the **devil curve**

$$\vec{r}(t) = [3t, t\sin(\pi t)\sin(t + \frac{1}{0.001 + t^2}), t\sin(\pi t)\cos(t + \frac{1}{0.001 + t^2})]$$

with $-1 \leq t \leq 1$. Note that we are close to a curve with discontinuities it was smoothed out. The curve looks evil, but it is not!



Solution:
This is a fundamental theorem of line integral problem. Find the function $f(x,y,z) = xyz + x^3/3 + y^3/3 + z^3/3$. Then get the points $B = r(1) = [3,0,0]$ and $A = r(-1) = [-3,0,0]$. Now evaluate $f(B) = f(r(1)) = 9$ and $f(A) = f(r(-1)) = -9$. The fundamental theorem tells that the line integral is $f(B) - f(A) = 18$.

Problem 11) (10 points)

What is the flux of the vector field

$$\vec{F} = [5x + yz, 3z - xz, xy + x + y + z]$$

through the surface S bounding the solid E seen in the picture? The solid E was obtained by taking a **hollow sphere**

$$9 < x^2 + y^2 + z^2 < 16$$

and slicing it into 20 equal pieces, then taking out every second, to half its volume. The surface is oriented outwards.

Solution:
This is solved with the divergence theorem. The divergence is constant 6. The result is 6 times the volume of E which is $6(4\pi 4^3/3 - 4\pi 3^3/3)/2 = 148\pi$.

Problem 12) (10 points)

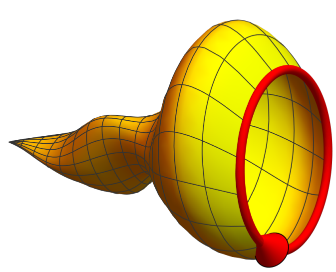
Let

$$\vec{F} = [z, y^2 e^x \cos(\cos(xyz)), -x]$$

be a vector field and S be the surface

$$\vec{r}(s,t) = [2s\cos(t)(1 - \frac{\sin(4\pi s)}{2}) + s\sin(2\pi s), 8s - 4, 2s\sin(t)(1 + \frac{\sin(3\pi s)}{2})]$$

seen in the picture, where $t \in [0, 2\pi]$ and $s \in [0, 1]$. The surface orientation is defined by the parametrization. We assure you that it is outwards. There is one boundary curve, obtained by putting $s = 1$. What is the flux of the curl of $\vec{F}$ through S?



Solution:
This is a Stokes theorem problem. Instead of the flux we compute the line integral. The line integral along the curve $r(t) = [2\cos(t), 0, 2\sin(t)]$ is $\int_0^{2\pi} [2\sin(t), 0, 2\cos(t)].[-2\sin(t), 0, 2\cos(t)] \; dt = -8\pi$. The orientation of the curve turned out to be ok.

Problem 13) (10 points)

We still remember the wonderful Toblerone chocolate eaten during the review. What is the area of the **Toblerone region** enclosed by the curve

$$\vec{r}(t) = [t^3 - t, 1 - t^4]$$

with $t \in [-1, 1]$?

Solution:
This is a Green's theorem problem. Take the vector field $\vec{F} = [0, x]$ and compute the line integral $\int_{-1}^{1} 1 - t^4(3t^2) \; dt = \int_{-1}^{1} 3t^2 - 3t^5 = 16/35$.