7/21/2022 SECOND HOURLY Practice 2 Maths 21a, O.Knill, Summer 2022

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F If $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$, then $D_{\vec{v}} f(0,0) = |\nabla f(0,0)|$.

Solution:

Angles dance upwards.

2) T F If $\vec{v} = \vec{r}\,'(0)$ is a unit vector and $\vec{r}(0) = \vec{0}$, then $\frac{d}{dt} f(\vec{r}(t))|_{t=0} = D_{\vec{v}} f(0,0)$.

Solution:

By definition

3) T F If $f(x,y,z) = 1$ defines a surface $z = g(x,y)$ with $g(1,1) = 2$ near $(x,y) = (1,1)$ then $g_x(1,1) = -f_x(1,1,2)/f_z(1,1,2)$.

Solution:

This is the implicit differentiation formula.

4) T F If $D_{[1,0]} f(1,1)$ is zero, then $(1,1)$ is a critical point of $f(x,y)$.

Solution:

This is only checking $f_x = 0$.

5) T F There exists a function $f(x,y)$ for which all points $x = y$ are critical points.

Solution:

like $f(x,y) = (x-y)^2$

6) T F The function $f(x,y) = y^2$ on the constraint $g(x,y) = x = 0$ has a global minimum.

Solution:

Just put $x = 0$ and see $-3y^2$.

7) T F If $(0,0)$ is a critical point of $f(x,y)$ satisfying $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) > 0$ and $f_{xy}(0,0) = 0$ then f has a minimum at $(0,0)$.

Solution:

Use the second derivative test.

8) T F The function $f(x,y) = x^2y - y$ has only one critical point.

Solution:

Look at the gradient. It is $[2xy, x^2 - 1]$ which gives $x = 1, -1, y = 0$.

9) T F If $\vec{r}(u,v)$ parametrizes a surface, then $2\vec{r}_u + 3\vec{r}_v$ is tangent to the surface.

Solution:

$\vec{r}_u$ and $\vec{r}_v$ are both tangent to the surface.

10) T F The surface area of a surface parametrized by $\vec{r}(u,v)$ over a domain R in the uv-plane is smaller or equal than $\int\int_R |\vec{r}_u \cdot \vec{r}_v| \, dudv$.

Solution:

We need $|\vec{r}_u \times \vec{r}_v|$.

11) T F If $f_{xx} = 0$ and $f_{xy} = 1$ for all (x,y) in the plane, then all critical points of f are saddle points.

Solution:

Yes, this implies $D < 0$.

12) T F For $\vec{u} = [0,1]$ and $\vec{v} = [1,0]$ the discriminant D in the second derivative test satisfies $D = (D_{\vec{u}}(D_{\vec{u}}f))(D_{\vec{v}}(D_{\vec{v}}f)) - (D_{\vec{u}}D_{\vec{v}}f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.

Solution:

This is the definition, if noticing that $f_u = D_u f$.

13) T F If $\vec{r}(t)$ parametrizes $f(x,y) = 1$, then the velocity vector $\vec{r}\,'(t)$ is perpendicular to $\nabla f(\vec{r}(t))$ for all t.

Solution:

By the gradient theorem

14) T F The function $f(x,y) = x^3y^2 + x^2y^3$ solves the PDE $f_{xyxyxy} = 0$.

Solution:

Use Clairaut's theorem.

15) T F If $f_x(x,y) = -f_y(x,y)$ for all x, y, then $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all (x,y).

Solution:

Differentiate f_x with respect to y to get f_{xy}. Differentiate f_y with respect to x to get f_{yx}. By Clairaut these are the same.

16) T F The linearization of $f(x,y) = xy$ at $(3,4)$ is $L(x,y) = 12+4(x-3)+3(y-4)$.

Solution:

The

17) T F If $f(x,t)$ solves the partial differential equation $f_t = f_{xx}$, then $g(x,t) = f(t,x)$ solves the partial differential equation $g_x = g_{tt}$.

Solution:

You differentiate twice

18) T **F** There is a function f such that $D_{\vec{u}} f(0,0)$ is negative for all unit vectors $\vec{u}$.

Solution:

If the function is positive in one direction, then it is negative in the opposite direction.

19) **T** F The surface area of the unit sphere does not depend on the parametrization.

Solution:

It is a general fact for parametrizations

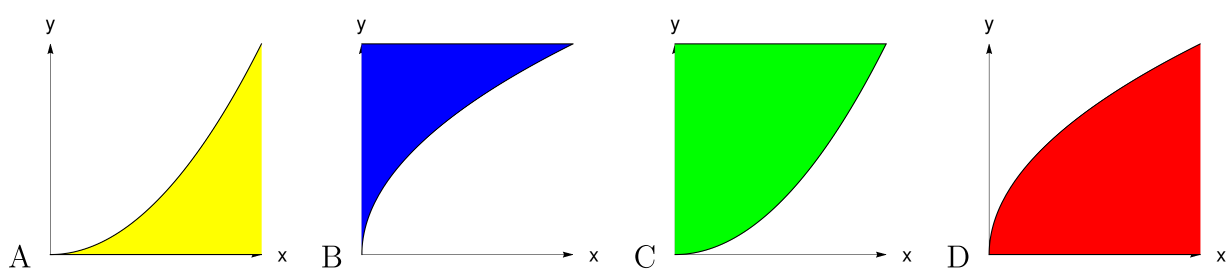
20) **T** F There is an f with a critical point with discriminant $D = 0$ and $f_{xx} > 0$ and $f_{xy} = 0$.

Solution:

$f = x^2 + y^4$ is possible.

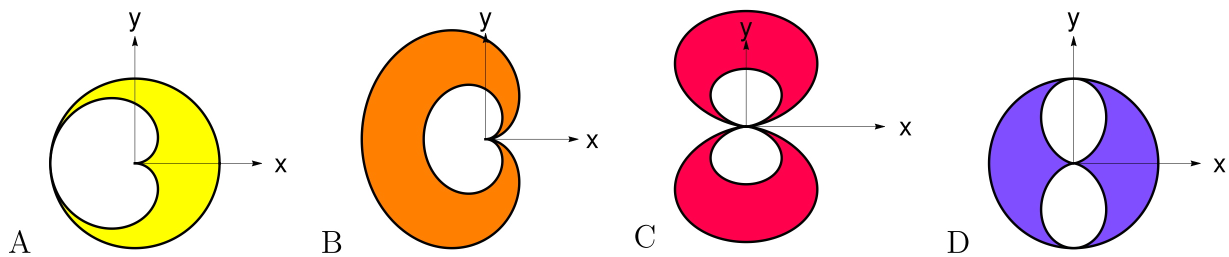
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with their area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{y^2}^1 1 \; dxdy$
	$\int_0^1 \int_0^{x^2} 1 \; dydx$
	$\int_0^1 \int_0^{\sqrt{y}} 1 \; dxdy$
	$\int_0^1 \int_{\sqrt{x}}^1 1 \; dydx$

b) (4 points) Match the regions with their area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^1 r \; drd\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^1 r \; drd\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r \; drd\theta$
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r \; drd\theta$

c) (2 points) Name all the four PDE's for a function $\psi(B, C)$ of variables B and C.

$\psi_B = \psi_C$	$\psi_{BB} + \psi_{CC} = 0$	$\psi_B + \psi\psi_C = \psi_{CC}$	$\psi_B = \psi - C\psi_C - C^2\psi_{CC}$

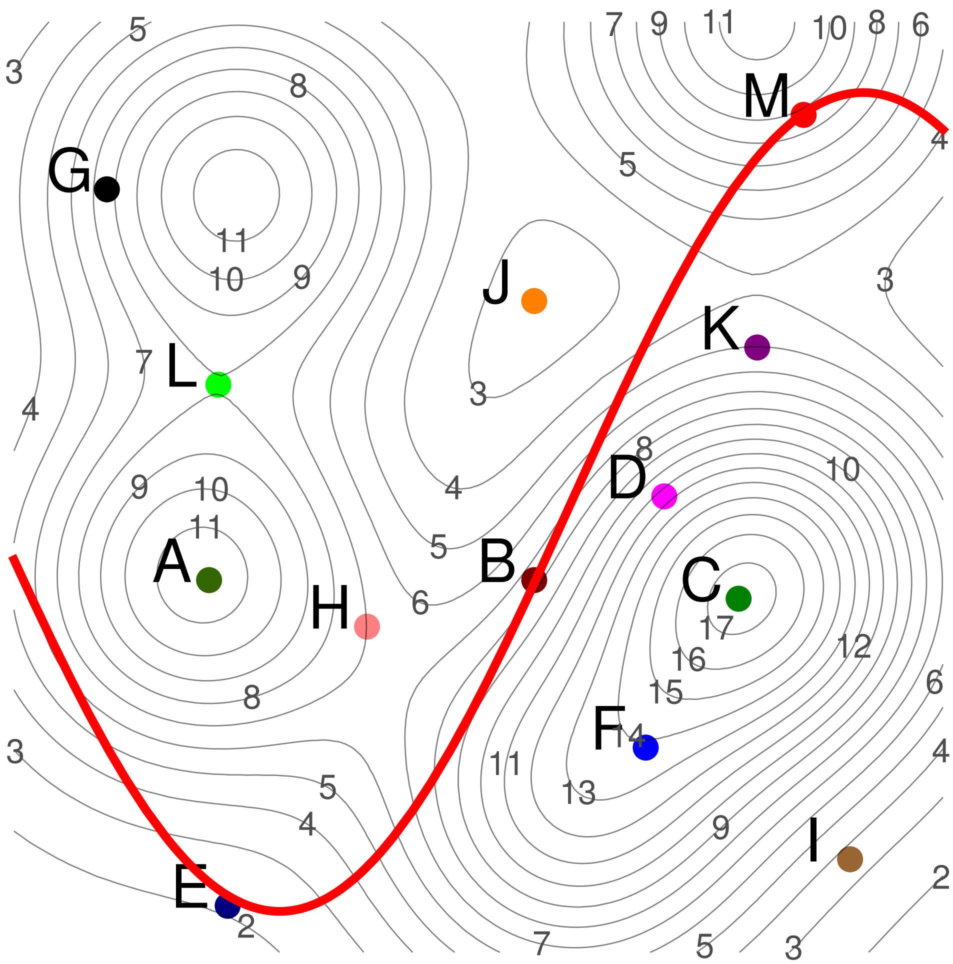
Solution:

a) DACB
b) DABC
c) Transport, Laplace, Burgers, Black Sholes

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We see the contours of an unknown smooth function, $f(x, y)$. The thick red curve is $g(x, y) = y - \sin(x) = 1$. Use each of the labels A-M only once. There are three labels which do not match.

	Enter A-M
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	
A saddle point of f.	
A local minimum of $f(x, y)$.	
A local but not global maximum of $f(x, y)$.	
A global maximum of $f(x, y)$.	
The point among A-M with maximal $\lvert\nabla f\rvert$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	



Solution:

G,F,K,L,J,A,C,D,M,E.

Problem 4) (10 points)

a) (8 points) Classify the critical points of the function

$$f(x,y) = 4x^2 - 4y^2 - 2x^4 + 8y$$

using the second derivative test.

Point	D	f_{xx}	nature

b) (2 points) Is there a global maximumn or minimum of $f(x,y)$? (No explanation necessary for this part b).)

	Yes	No
There is a global max for f		
There is a global min for f		

Solution:

a)

x	y	D	$f_x x$	Type	f
-1	1	128	-16	maximum	6
0	1	-64	8	saddle	4
1	1	128	-16	maximum	6

b) Global Max but no global min.

Problem 5) (10 points)

Find the minimum of the function

$$f(x,y) = 5 + x^2 + y^2 + 2xy$$

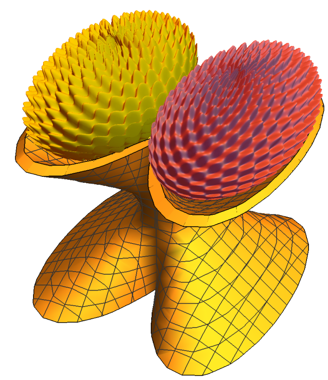
under the constraint $x + 2y = 5$. You need to use the Lagrange method.

$(x,y) =$

Solution:

The Lagrange equations lead to $x = -y$. The solution is $(-5, 5)$

Problem 6) (10 points)



Ice cream sponsored by math-candy.com.

a) 5 points) Find the equation $ax + by + cz = d$ of the tangent plane to the surface

$$f(x, y, z) = x^4 + y^4 + z^2 + x^2y^2 - x^2z^2 + y^2 - z^2 = 3$$

at the point $(1, 1, 1)$.

Plane

b) (5 points) Estimate $f(1.01, 1.02, 0.97)$ using linearization.

Estimate

Solution:

a) $2x + 4y - z = 5$.
b) $3 + 4 * 0.01 + 8 * -.02 + (-2) * (-0.03) = 3.26$.

Problem 7) (10 points)

Find the surface area of the surface

$$\vec{r}(u, v) = [\frac{v^2}{2} - 3, \frac{u^2}{2} + 3, \frac{uv}{\sqrt{2}}] .$$

for which the parameters satisfy $u^2 + v^2 \leq 4$.

Surface area

Solution:

$$|\vec{r}_u \times \vec{r}_v| = (u^2 + v^2)/\sqrt{2}$$

To integrate over the parameter region, we use polar coordinates:

$$\int_0^{2\pi} \int_0^2 r^2/\sqrt{2}\ r\ dr d\theta\ = 4\sqrt{2}\pi = 8\pi/\sqrt{2}\ .$$

Problem 8) (10 points)

Assume we know $f_x(3,3) = 1$ and $D_{[1,1]/\sqrt{2}} f(3,3) = 2\sqrt{2}$.

a) (5 points) Find the tangent line to the curve $\{f(x,y) = 7\}$ at $(3,3)$.

Tangent line

b) (5 points) Estimate $f(3.0001, 3.003)$ using linearization.

Estimate

Solution:

a) $f_x = 1, f_y = 3,\ x + 3y = 12$
b) $7 + 0.0001 + 0.009 = 7.0091.$

Problem 9) (10 points)

a) (5 points) Evaluate the following double integral

$$\iint_G (x^2 + y^2)^4 \; dxdy \;,$$

where G is region given by

$$\{x^2 + y^2 \leq 4, x \geq 0, y \geq 0\} \;.$$

Result

b) (5 points)

$$\int_0^1 \int_0^{\arctan(x)} \frac{1}{1 - \tan(y)} \; dydx \;.$$

Result

Solution:

a) $\int_{\pi/2} \int_0^2 r^8 \cdot r \; dr d\theta = 2^{10}\pi/20 = 256\pi/5.$
b) Switch the order of integration. To do so, make a picture.

$$\int_0^{\pi/4} \int_1^{\tan(y)} \frac{1}{1 - \tan(y)} dxdy = \int_0^{\pi/4} 1 \; dy = \frac{\pi}{4} \;.$$