7/21/2022 SECOND HOURLY **Maths 21a, O.Knill, Summer 2022**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The integral $\iint_R 1 \, dA$ is the area of a two dimensional region R.

2) T F If $x^4 y + y^3 x = 2$ defines y as a function of x then by implicit differentiation, $y'(1) = -5/4$.

3) T F If $f_{xx} > 0$ and the discriminant $D > 0$, then $f_{yy} < 0$.

4) T F $(0, 0)$ is neither a max nor min of the function $f(x, y) = x^4 + y^4$ because the discriminant D is zero there.

5) T F If $R = \{x^2 + y^2 \leq 9\}$ then $\iint_R x^2 + y^2 \, dxdy = \int_0^{2\pi} \int_0^3 r^2 \, dr d\theta$.

6) T F The value of the function $f(x, y) = \sin(2x + 4y)$ at $(x, y) = (0.01, -0.01)$ can by linear approximation near $(x_0, y_0) = (0, 0)$ be estimated as -0.02.

7) T F If $(1, 1)$ is a critical point of $f(x, y)$, then $(1, 1)$ is also a critical point for the function $g(x, y) = f(x^3, y)$.

8) T F The gradient of $f(x, y)$ is normal to the level curves of f.

9) T F If (x_0, y_0) is a max of $f(x, y)$ under the constraint $g(x, y) = g(x_0, y_0)$, then (x_0, y_0) is a max of $g(x, y)$ under some constraint $f(x, y) = f(x_0, y_0)$.

10) T F The area of a filled ellipse $x^2/4 + y^2/9 \leq 4$ is equal to 4 times the area of the filled ellipse $x^2/4 + y^2/9 \leq 1$.

11) T F If $\vec{v}$ is a unit vector parallel to the surface $f(x, y, z) = 0$ $D_{\vec{v}} f(x, y, z) = 0$.

12) T F If $\vec{r}(t) = [x(t), y(t)]$ and $x(t), y(t)$ are non-constant polynomials like $x(t) = 1 + t^2, y(t) = 1 + t^3$ then the unit tangent vector is defined at all points.

13) T F The vector $\vec{r}_v(u, v)$ is tangent to the surface parameterized by $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$.

14) T F The second derivative test involving D and f_{xx} can check whether a Lagrange multiplier solution of f under the constraint $g = c$ is a max or min.

15) T F If $(0, 0)$ is a critical point of $f(x, y)$ and $D = 0$ but $f_{xx}(0, 0) > 0$ then $(0, 0)$ is not a local max.

16) T F Let (x_0, y_0) be a saddle point of $f(x, y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

17) T F If $f(x, y)$ has two max, then it must have a min.

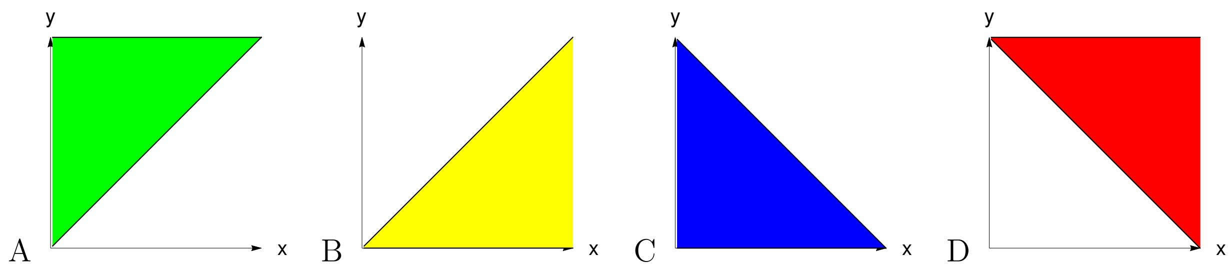
18) T F Given a unit vector $\vec{v}$ and (x_0, y_0) a critical point of f, then $D_{\vec{v}} f(x_0, y_0) > 0$ at a local minimum.

19) T F The chain rule assures that $d/dt(x(t)^2 + y(t)^2)) = 2x(t)x'(t) + 2y(t)y'(t)$.

20) T F The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

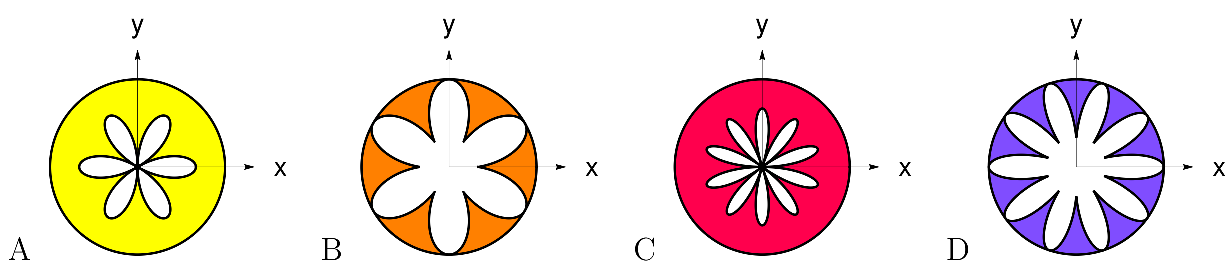
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_y^1 1 \; dxdy$
	$\int_0^1 \int_0^{1-x} 1 \; dydx$
	$\int_0^1 \int_0^y 1 \; dxdy$
	$\int_0^1 \int_{1-x}^1 1 \; dydx$

b) (4 points) Now match polar regions with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{1+2\vert\sin(3\theta)\vert}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{1+2\vert\cos(5\theta)\vert}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{2\vert\sin(5\theta)\vert}^3 r dr d\theta$
	$\int_0^{2\pi} \int_{2\vert\cos(3\theta)\vert}^3 r dr d\theta$

c) (2 points) Write down the differential equations for the unknown function $f(t,x)$.

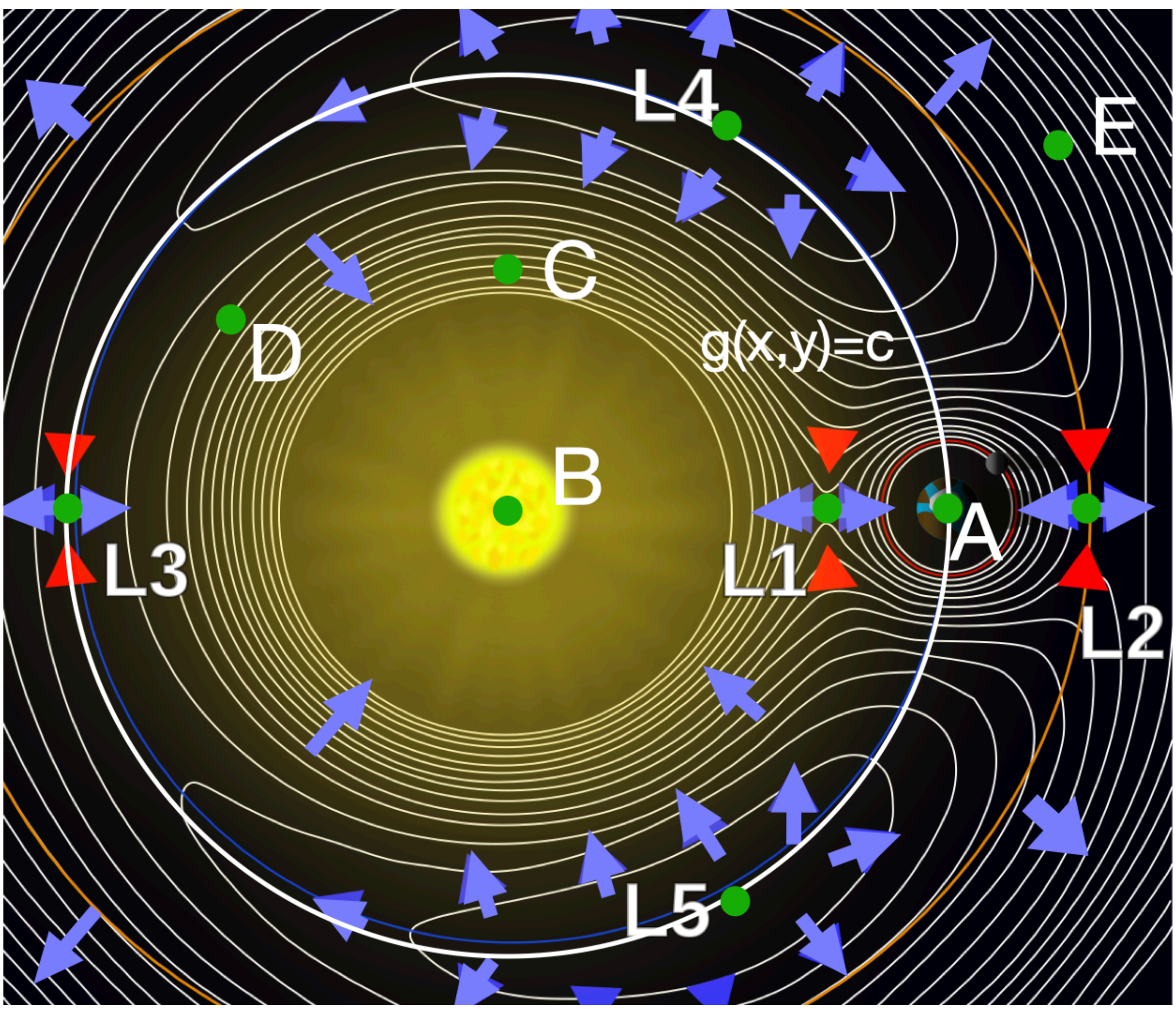
Transport equation:

Heat equation:

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We all have seen the nice pictures of the **Webb telescope**. It sits at the **Lagrange point** L_2 on the line through the sun (B) and earth (A). We modified a Wikipedia picture explaining the Lagrange points and added a few more points $A-E$. The contour map visualizes the effective potential of the rotating system in which the sun-earth axis is at rest. The vectors you see represent gradients $\nabla f(x,y)$. The circle through A, L_3, L_4, L_5 centered at the sun B is written here as $g(x,y) = c$. Each of the questions is worth 1 point.

	Enter A-E,L_1,...,L_5 here
Which points are minima of f?	
Which points are maxima of f?	
Which points are saddles?	
Point with maximal $\|\nabla f\|$.	
Points with $f_x > 0$.	
Points with $f_x < 0$.	
Points with $f_y > 0$.	
Points with $f_y < 0$.	
Points, where f is maximal on $g(x,y) = c$.	
Points, where f is minimal on $g(x,y) = c$.	



Problem 4) (10 points)

Classify the critical points of the function

$$f(x,y) = x^3y - xy$$

using the **second derivative test**. Is there a global max or min?

Problem 5) (10 points)

Use the Lagrange method to solve the problem to minimize

$$f(x,y) = x^2 + y^2 + xy$$

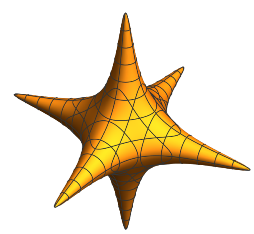
under the constraint $g(x,y) = 3x + 4y = 26$.

Problem 6) (10 points)

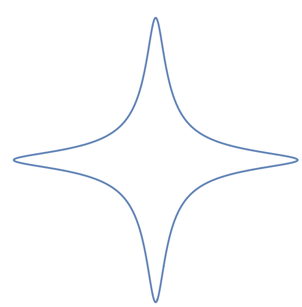
a) (5 points) When we look at the contour surface

$f(x,y,z) = 10x^2y^2+10y^2z^2+10x^2z^2+(x^2+y^2+z^2) = 25$,

we see some sharp spikes near the coordinate axes. To find out whether there is a singularity there, we compute the **tangent plane** at the point $(x_0, y_0, z_0) = (5, 0, 0)$.



b) (5 points) To understand this better, we cut up the surface at $z = 0$ and have a level curve $g(x,y) = 10x^2y^2 + x^2 + y^2 = 25$. Now find the **tangent line** $ax + by = d$ to this curve at the point $(x_0, y_0) = (5, 0)$.



Problem 7) (10 points)

On the last winter trip to **Martha's Vineyard**, "Olli Rocky Docky" and family took a steam boat. Olli uses a penny pressing machine there to deform a penny into a surface

$$\vec{r}(u,v) = [uv, u-v, u+v]$$

with parameters satisfying $0 \leq u^2 + v^2 \leq 9$. Find the **surface area** of the surface. (Only one side of the penny of course).

Problem 8) (10 points)

For more than a decade, there has been a "gold rush" in **Madagascar**. It is not gold but **sapphires** which attract the miners. Assume $f(x,y,z)$ is the probability to find a blue saphire. You know $f_z(1,2,3) = 1$ and measure

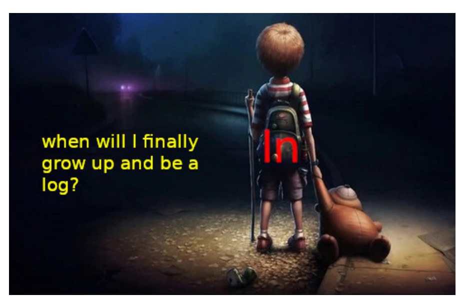
$$D_{\vec{v}} f(1,2,3) = 11/\sqrt{3}$$

if you go into the direction $\vec{v} = [1,1,1]/\sqrt{3}$. You also experience

$$D_{\vec{w}} f(1,2,3) = 7$$

for $\vec{w} = [3,4,0]/5$. To see how your luck changes when digging into the direction $\vec{u} = [1,-1,1]/\sqrt{3}$, compute the directional derivative $D_{\vec{u}} f(1,2,3)$.

Problem 9) (10 points)



a) (5 points) Evaluate the double integral

$$\int_1^e \int_{\log(x)}^1 \frac{y}{e^y - 1} \, dy dx ,$$

where log is the natural log as usual.

b) (5 points) Evaluate the double integral

$$\iint_G \log(x^2 + y^2) \, dxdy ,$$

where G is region given by

$$\{4 \leq x^2 + y^2 \leq 9\} .$$

P.S. as a grown-up, you also know that integration by parts gives $\int \log(x) \, dx = x\log(x) - x$.