

### Triple Integrals

- $\iiint_R f(x, y, z) dzdydx$  triple integral
- $\int_a^b \int_c^d \int_u^v f(x, y, z) dzdydx$  integral over rectangular box
- $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y) dzdydx$  type I region
- $\iiint_R f(r, \theta, z) \boxed{r} dzdrd\theta$  integral in cylindrical coordinates
- $\iiint_R f(\rho, \theta, \phi) \boxed{\rho^2 \sin(\phi)} d\rho d\phi d\theta$  integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x, y, z) dzdydx = \int_u^v \int_c^d \int_a^b f(x, y, z) dx dy dz$  Fubini
- $V = \iiint_E \boxed{1} dzdydx$  volume of solid  $E$

### Line Integrals

- $\vec{F}(x, y) = [P(x, y), Q(x, y)], \vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]$  vector field.
- $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$  line integral
- $\vec{F}(x, y) = \nabla f(x, y)$  gradient field = potential field = conservative field

### Fundamental theorem of line integrals

- FTL:  $\vec{F}(x, y) = \nabla f(x, y), \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = f(\vec{r}(b)) - f(\vec{r}(a))$
- closed loop property  $\int_C \vec{F} \cdot d\vec{r} = 0$ , for all closed curves  $C$
- always equivalent: closed loop property, path independence and gradient field
- Clairaut test  $\text{curl}(\vec{F}) \neq 0$  assures  $\vec{F}$  is not a gradient field

### Green's Theorem

- $\vec{F}(x, y) = [P, Q]$ , curl in two dimensions:  $\text{curl}(\vec{F}) = Q_x - P_y$
- Green's theorem:  $C$  boundary of  $R$ , then  $\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(\vec{F}) dx dy$
- Take  $\vec{F} = [-y, 0]$  or  $\vec{F} = [0, x]$  to get area
- Green's theorem is useful to compute difficult line integrals or difficult 2D integrals

### Flux integrals

- $\vec{F}(x, y, z)$  vector field,  $S = \vec{r}(R)$  parametrized surface
- $\vec{r}_u \times \vec{r}_v$  normal vector,  $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$  unit normal vector
- $\vec{r}_u \times \vec{r}_v du dv = d\vec{S} = \vec{n} dS$  normal surface element
- $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$  flux integral

### Stokes Theorem

- $\vec{F}(x, y, z) = [P, Q, R], \text{curl}([P, Q, R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = \nabla \times \vec{F}$
- Stokes's theorem:  $C$  boundary of surface  $S$ , then  $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$
- Stokes theorem allows to compute difficult flux integrals or difficult line integrals

## Grad Curl Div

- $\nabla = [\partial_x, \partial_y, \partial_z], \vec{F} = \nabla f, \text{curl}(\vec{F}) = \nabla \times \vec{F}, \text{div}(\vec{F}) = \nabla \cdot \vec{F}$
- $\text{div}(\text{curl}(\vec{F})) = 0$  and  $\text{curl}(\text{grad}(f)) = \vec{0}$
- $\text{div}(\text{grad}(f)) = \Delta f$  Laplacian
- incompressible = divergence free field:  $\text{div}(\vec{F}) = 0$  everywhere. Implies  $\vec{F} = \text{curl}(\vec{G})$
- irrotational =  $\text{curl}(\vec{F}) = 0$  everywhere. Implies  $\vec{F} = \text{grad}(f)$

## Divergence Theorem

- $\text{div}([P, Q, R]) = P_x + Q_y + R_z = \nabla \cdot \vec{F}$
- divergence theorem: solid  $E$ , boundary  $S$  then  $\iint_S \vec{F} \cdot dS = \iiint_E \text{div}(\vec{F}) dV$
- the divergence theorem allows to compute difficult flux integrals or difficult 3D integrals

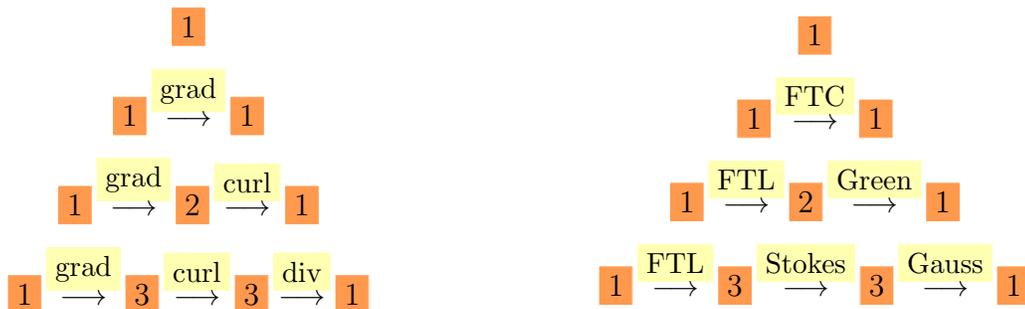
## Some topology

- interior of a region  $D$ : points in  $D$  for which small neighborhood is still in  $D$
- boundary of a curve: the end points of the curve if they exist
- boundary of a surface  $S$  points on surface belonging to parameters  $(u, v)$  not in the interior of the parameter domain
- boundary of a solid  $D$ : the surfaces which bound the solid, points in the solid which are not in the interior of  $D$
- closed surface: a surface without boundary like for example the sphere
- closed curve: a curve with no boundary like for example a knot

## Integration overview

- Line integral:  $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$
- Flux integral:  $\iint_S \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) dudv$

## Theorems and Derivatives overview



# Integral Theorems Overview

## INTEGRATION.

- Line integral:**  $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$
- Flux integral:**  $\iint_R \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v dudv$
- Double integral:**  $\iint_R f(x, y) dx dy.$
- Triple integral:**  $\iiint_E f(x, y, z) dx dy dz.$

- Area**  $\iint_R 1 dA = \iint_R 1 dx dy$
- Arc length**  $\int_a^b |\vec{r}'(t)| dt$
- Surface area**  $\iint \sqrt{|\vec{r}_u \times \vec{r}_v|} dudv$
- Volume**  $\iiint_B 1 dx dy dz$

## DIFFERENTIATION.

- Velocity:**  $\vec{r}'(t) = \frac{d}{dt} \vec{r}(t).$
- Partial derivative:**  $f_x(x, y, z).$
- Gradient:**  $\text{grad}(f) = [f_x, f_y, f_z]$
- Curl in 2D:**  $\text{curl}([P, Q]) = Q_x - P_y$
- Curl in 3D:**  $\text{curl}[P, Q, R] = [R_y - Q_z, P_z - R_x, Q_x - P_y]$
- Div:**  $\text{div}(\vec{F}) = \text{div}[P, Q, R] = P_x + Q_y + R_z.$

## IDENTITIES.

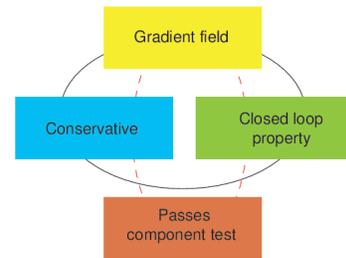
- $\text{div}(\text{curl}(\vec{F})) = 0$
- $\text{curl}(\text{grad}(f)) = \vec{0}$
- $\text{div}(\text{grad}(f)) = \Delta f.$

## JARGON.

- $\text{div}(\vec{F}) = 0$  incompressible
- $\text{curl}(\vec{F}) = \vec{0}$  irrotational

## CONSERVATIVE FIELDS:

- Gradient fields:  $\vec{F} = \text{grad}(f).$
- Closed curve property:  $\int_C \vec{F} \cdot d\vec{r} = 0$  for any closed curve.
- Conservative:  $C_1, C_2$  paths from  $A$  to  $B$ , then  $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$
- Mixed derivative test:  $\text{curl}(F) = 0$  if  $\vec{F} = \nabla f.$
- If  $\vec{F}$  is irrotational, then  $\vec{F} = \nabla f$  in simply connected  $R.$



## TOPOLOGY.

- Interior** of region  $D$ : points which have a neighborhood contained in  $D.$
- Boundary** of curve: endpoints. **Boundary** of surface: curves. **Boundary** of solid: surfaces.
- Simply connected:** a closed curve in  $R$  can be deformed inside  $R$  to a point.
- Closed curve** Curve without boundary.
- Closed surface** surface without boundary.

LINE INTEGRAL THEOREM. If  $C : \vec{r}(t) = [x(t), y(t), z(t)], t \in [a, b]$  is a curve and  $f(x, y, z)$  is a function.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

GREEN'S THEOREM. If  $R$  is a region with boundary  $C$  and  $\vec{F} = [P, Q]$  is a vector field, then

$$\iint_R \text{curl}(F) dx dy = \int_C \vec{F} \cdot d\vec{r}$$

STOKES THEOREM. If  $S$  is a surface with boundary  $C$  and  $\vec{F}$  is a vector field, then

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

DIVERGENCE THEOREM. If  $S$  is the boundary of a solid  $E$  in space with boundary surface  $S$  and  $\vec{F}$  is a vector field, then

$$\iiint_E \operatorname{div}(\vec{F}) \, dV = \iint_S \vec{F} \cdot d\vec{S}$$

GENERAL STOKES. All theorems are of the form

$$\int_G dF = \int_{\delta G} F$$

where  $dF$  is the derivative of  $F$  and  $\delta G$  is the oriented boundary of  $G$ .



George Gabriel Stokes



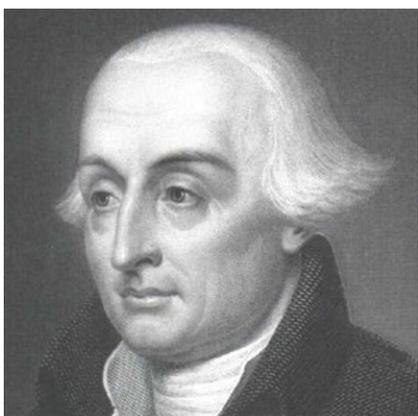
Mikhail Ostrogradsky



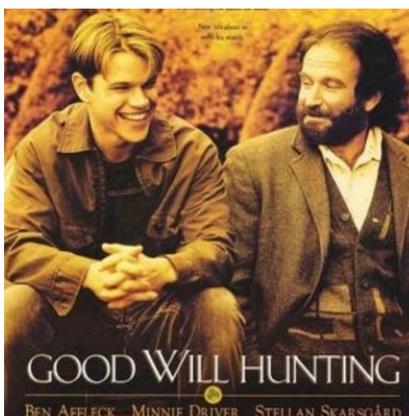
Carl Friedrich Gauss



André-Marie Ampère



Joseph Louis Lagrange



George Green