Partial Derivatives

tangent line: $ax + by = d$ with $a = f_x(x_0, y_0), b = f_y(x_0, y_0), d = ax_0 + by_0$
tangent plane: $ax + by + cz = d$ with $a = f_x, b = f_y, c = f_z, d = ax_0 + by_0 + cz_0$
$f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
$L(x, y) = f(x_0, y_0) + a(x - x_0) + b(y - y_0)$ linear approximation
$L(x, y)$ estimates $f(x, y)$ near $f(x_0, y_0)$. The result is $f(x_0, y_0) + a(x - x_0) + b(y - y_0)$
estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
$f_{xy} = f_{yx}$ Clairaut's theorem, if f_{xy} and f_{yx} are continuous.
$\vec{r}_u(u, v), \vec{r}_v(u, v)$ tangent to surface parameterized by $\vec{r}(u, v)$

Partial Differential Equations

$f_t = f_{xx}$ heat equation
$f_{tt} - f_{xx} = 0$ wave equation
$f_x - f_t = 0$ transport equation
$f_{xx} + f_{yy} = 0$ Laplace equation
$f_t + f f_x = f_{xx}$ Burgers equation
$f_x^2 + f_y^2 = 1$ Eiconal equation
$f_t = f - x f_x - x^2 f_{xx}$ Black Scholes
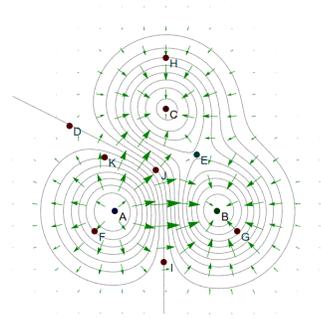
Gradient

$\nabla f(x, y) = [f_x, f_y]$, $\nabla f(x, y, z) = [f_x, f_y, f_z]$, gradient
a vector of length 1 is a **direction** also called unit vector
$D_{\vec{v}} f = \nabla f \cdot \vec{v}$ directional derivative, $\vec{v}$ is direction
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ chain rule
$\nabla f(x_0, y_0)$ is orthogonal to the level curve $f(x, y) = c$ containing (x_0, y_0)
$\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ containing (x_0, y_0, z_0)
$\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_{\vec{v}} f$ by chain rule
$(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) = 0$ tangent line
$(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane
directional derivative at (x_0, y_0) is maximal in the $\vec{v} = \nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$ direction
angels dance upwards: $f(x, y)$ increases in the $\nabla f/|\nabla f|$ direction if $|\nabla f| \neq 0$
partial derivatives are special directional derivatives: $D_{\vec{i}} f = f_x$
if $D_{\vec{v}} f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$
$f(x, y, z) = c$ defines $z = g(x, y)$, and $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$ implicit diff

Extrema

$\nabla f(x, y) = [0, 0]$, critical point
$D = f_{xx} f_{yy} - f_{xy}^2$ discriminant as used in second derivative test
$f(x_0, y_0) \geq f(x, y)$ in a neighborhood of (x_0, y_0) local maximum
$f(x_0, y_0) \leq f(x, y)$ in a neighborhood of (x_0, y_0) local minimum
$\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = c$, λ Lagrange equations
$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = c$, λ Lagrange equations

2. deriv. test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**
2. deriv. test: $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**
2. deriv. test: $\nabla f = (0,0), D < 0$ **saddle point**
$f(x_0, y_0) \geq f(x, y)$ everywhere, global maximum
$f(x_0, y_0) \leq f(x, y)$ everywhere, global minimum
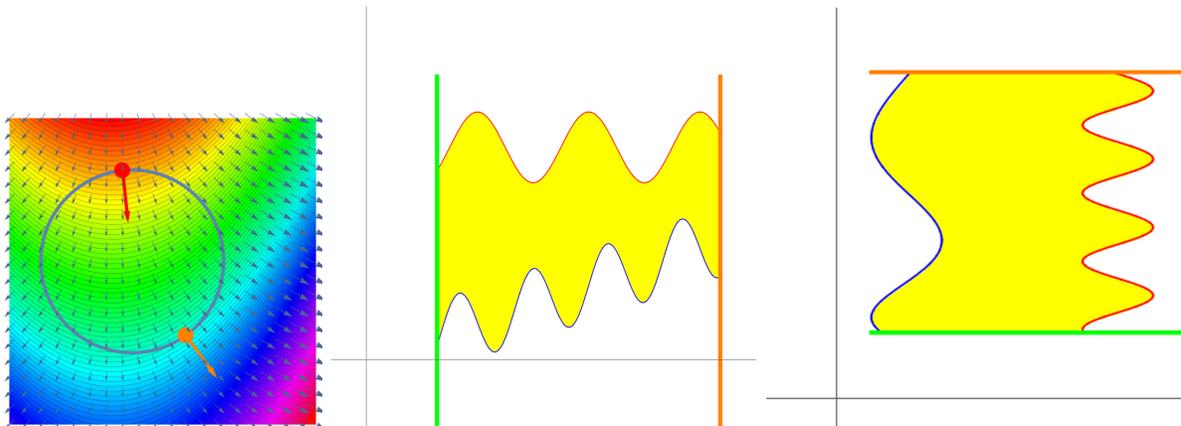
Double Integrals

$\iint_R f(x, y)\, dydx$ double integral
$\int_a^b \int_c^d f(x, y)\, dydx$ integral over rectangle
$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dydx$ bottom-to-top region
$\int_c^d \int_{a(y)}^{b(y)} f(x, y)\, dxdy$ left-to-right region
$\iint_R f(r, \theta)\, \boxed{r}\, drd\theta$ polar coordinates
$\int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy$ Fubini
$\iint_R \boxed{1}\, dxdy$ area of region R
$\iint_R f(x, y)\, dxdy$ signed volume of solid bound by graph of f and xy-plane

Surface Area

$\iint_R |\vec{r}_u \times \vec{r}_v|\, dudv$ surface area
$\iint_R \sqrt{1 + f_x^2 + f_y^2}; dudv$ surface area of graph $z = f(x, y)$.
$2\pi \int_a^b g(z) \sqrt{1 + g'(z)^2}\, dz$ surface area of $z = g(z), a \leq z \leq b$.

General advise

Draw the region when integrating in in higher dimensions.
Consider other coordinate systems if the integral does not work.
Consider changing the order of integration if the integral does not work.
If an integral gets too complex, track back your steps to spot errors.
For tangent planes, compute the gradient $[a, b, c]$ first then worry about the constant.
When looking at relief problems, mind the gradient. It is the key!

Oliver Knill, last update: July 18, 2022