MULTIVARIABLE CALCULUS

MATH S-21A

Unit 16: Surface Integration

Lecture

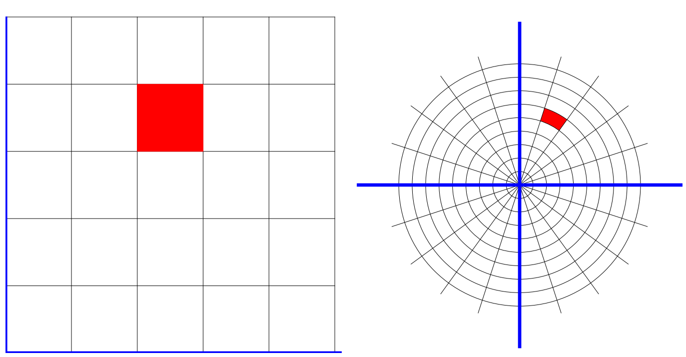
16.1. For certain regions, it is better to use a different coordinate system. A re-parametrization $(x, y) = \vec{r}(u, v)$ often helps. This works then also in higher dimensions, where surfaces are parametrized as $[x, y, z] = \vec{r}(u, v)$. We first remain in $\mathbb{R}^2$, where polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta)$ are an important example.

Definition: A **polar region** is a planar region bound by a simple closed curve. It is defined in polar coordinates by a curve $(t, r(t))$ where $t = \theta$ is the angle. In Cartesian coordinates, the parametrization of the boundary of a polar region is $\vec{r}(t) = [r(t)\cos(t), r(t)\sin(t)]$, a **polar graph** like the spiral with $r(t) = t$.

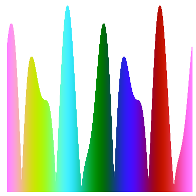
Theorem: To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x, y)\ dxdy = \iint_R f(r\cos(\theta), r\sin(\theta))r\ drd\theta\ .$$

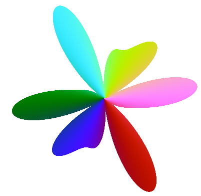
16.2. Why do we have to include the factor r, when we transition to polar coordinates? The reason is that a small rectangle R with area $dA = d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a **sector segment** S in the (x, y) plane. It has the area $r\ d\theta dr$. We will also see that the parametrization $\vec{r}(\theta, r) = [r\cos(\theta), r\sin(\theta), 0]$ gives $|\vec{r}_\theta \times \vec{r}_r| = r$.



16.3. We can now integrate over basic regions in the (θ, r)-plane. Examples are **flowers**: $\{(\theta, r) \,|0 \leq r \leq f(\theta)\}$, where $f(\theta)$ is a non-negative periodic function of θ.



A polar region shown in polar coordinates.

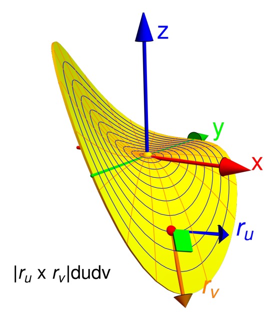


The same region in the xy coordinate system.

Theorem: A surface $\vec{r}(u, v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv \; .$$

16.4. Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u, v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u, v)$. The two vectors span a **parallelogram** with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u + du] \times [v, v + dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $\vec{r}_u du$ and $\vec{r}_v dv$ which has the area $|\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv$.

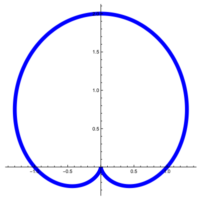
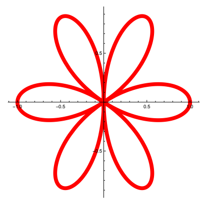
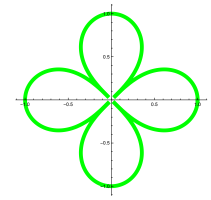


Examples

16.5. The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**. Note that in the literature you often see also situations where $r(\theta)$ can become negative. We will never allow that as $r \geq 0$ is a radius and a radius is non-negative in order not to get confused.

16.6. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a **heart** and belongs to the class of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

16.7. The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**.

16.8. Integrate

$$f(x,y) = x^2 + y^2 + xy$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\; dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r \; d\theta dr = 2\pi/4$.

16.9. We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\}$ using substitution. It is more elegant to do this in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r \; drd\theta = 2\pi r^2/2|_0^1 = \pi \; .$$

16.10. Integrate the function $f(x,y) = 1$ $\{(\theta, r(\theta)) \mid r(\theta) \leq \; |\cos(3\theta)|\}$.

$$\int \int_R 1 \; dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r \; dr \; d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} \; d\theta = \pi/2 \; .$$

16.11. Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y) \mid 1 < x^2 + y^2 < 4, y > 0\}$.

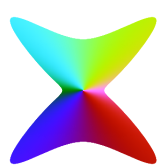
$$\int_1^2 \int_0^{\pi} r\sin(\theta) r \; r \; d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta) \; d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta) \; d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$, always first try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

16.12. The Belgian Biologist **Johan Gielis** defined in 1997 the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, squares, triangles or stars. It can also be used to produce “super-shapes”. The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]

[1]”Gielis, J. A ’generic geometric transformation that unifies a wide range of natural and abstract shapes’. American Journal of Botany, 90, 333 - 338, (2003).

16.13. The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]$ is part of the xy-plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]$ and we see that the area of $S = \vec{r}(R)$ is 6 times the area of R.

16.14. The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]$ maps the rectangle $G = \{0 \leq x \leq 2\pi, 0 \leq y \leq \pi$ onto a sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L^2\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1\ dS = \int_0^{2\pi}\int_0^{\pi} L^2\sin(v)\ dvdu = 4\pi L^2$. This is a formula which Archimedes already has derived by seeing it as the surface area of an open cylinder of height $2L$ and radius L.

16.15. For graphs $(u,v) \mapsto [u, v, f(u,v)]$, we have $\vec{r}_u = [1, 0, f_u(u,v)]$ and $\vec{r}_v = [0, 1, f_v(u,v)]$. The cross product $\vec{r}_u \times \vec{r}_v = [-f_u, -f_v, 1]$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region R is $\int\int_R \sqrt{1 + f_u^2 + f_v^2}\ dudv$.

16.16. Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]$ on $R = [0, 2\pi] \times [a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v|\ dudv = 2\pi \int_a^b |f(v)|\sqrt{1 + f'(v)^2}\ dv$.

Homework

This homework is due on Tuesday, 7/19/2022.

Problem 16.1: Find $\int\int_R (x^2 + y^2)^{150}\ dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

Problem 16.2: The **cardioid** has first been described in 1741. Its ultimate fame came because the **main body of the Mandelbrot** set is a Cardioid. Find its area. Its boundary is

$$\vec{r}(t) = [\frac{\cos(t)}{2} - \frac{\cos(2t)}{4}, \frac{\sin(t)}{2} - \frac{\sin(2t)}{4}], 0 \leq t \leq 2\pi\ .$$

Problem 16.3: Find the area of the region bounded by three curves: first by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 3\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x-axis?

Problem 16.4: Find the average $\frac{\iint_R f\ dxdy}{\iint_R 1\ dxdy}$ for $f(x,y) = 34(x^2 + y^2)$ on the annular region $R : 1 \leq |(x,y)| \leq 2$.

Problem 16.5: Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 16$.