MULTIVARIABLE CALCULUS

MATH S-21A

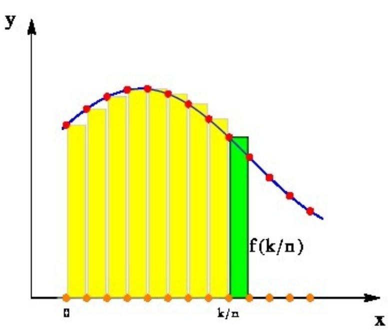
Unit 15: Double Integrals

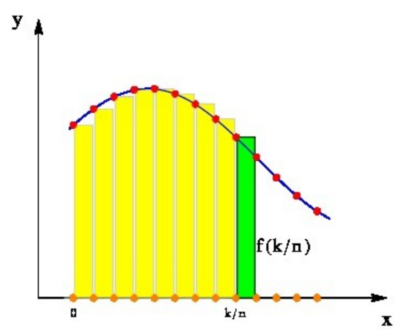
Lecture

15.1. If $f(x)$ is a continuous function of one variable, then the **Riemann integral** $\int_a^b f(x)\ dx$ is defined as the limit of the **Riemann sums** $S_n f = \frac{1}{n} \sum_{k/n \in [a,b]} f(k/n)$ for $n \to \infty$. The **derivative** of f is the limit of **difference quotients** $D_n f(x) = n[f(x+1/n) - f(x)]$ as $n \to \infty$. The **integral** $\int_a^b f(x)\ dx$ is the **signed area** under the graph of f and above the x-axes, where "signed" indicates that area below the x-axes has negative sign. The function $F(x) = \int_0^x f(y)\ dy$ is called an **anti-derivative** of f. It is determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x)\ dx = F(x) - F(0)\ .$$

It allows to compute integrals by inverting differentiation. **Differentiation rules** so become **integration rules**: the product rule leads to integration by parts; the chain rule becomes substitution.





Definition: If $f(x,y)$ is continuous on a region R, the integral $\iint_R f(x,y)\ dxdy$ is defined as the limit of Riemann sums

$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n}) \in R} f(\frac{i}{n}, \frac{j}{n})$$

when $n \to \infty$. We write also $\iint_R f(x,y)\ dA$, where $dA = dxdy$ is a formal notation meaning for "an area element".

15.2. The **Fubini's theorem** allows to switch the order of integration over a rectangle if the function f is continuous:

Theorem: $\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$.

Proof. For every n, there is the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

which holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$. This is possible for continuous functions. Fubini's theorem only holds for rectangles. We extend now the collection of possible regions to integrate over:

Definition: A **bottom to top region** is of the form

$$R = \{(x,y) \mid a \leq x \leq b,\ c(x) \leq y \leq d(x)\ \} .$$

An integral over a **bottom to top region** is called a **bottom to top integral**

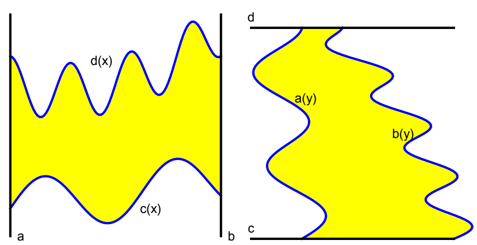
$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx .$$

A **left to right region** is of the form

$$R = \{(x,y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\ \} .$$

An integral over such a region is called a **left to right integral**

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



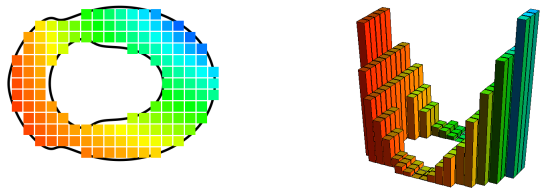
15.3. Similarly as we could see in one dimensions, an integral as a signed area, one can interpret $\int\int_R f(x,y)\ dydx$ as the **signed volume** of the solid below the graph of f and above R in the xy plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

Examples

15.4. If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how to reduce double integrals to single variable integrals.

15.5. If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ contained in R.

15.6. The value $\iint_R f(x,y)\,dA / \iint_R 1\,dA$ is the **average** value of f.



15.7. Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \leq x \leq \pi^{1/3}$. The value of this integral has a physical meaning. It is called **moment of inertia**.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\, dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\, dx$$

We have now an integral, which we can solve by substitution $-\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3}$.

15.8. Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved with a "left to right" integral. As you can see from the picture, we would have to compute two different integrals as a "bottom to top" integral. To do so, we have to write the bounds as a function of x: they are $x = y-1$ and $x = 1-y$

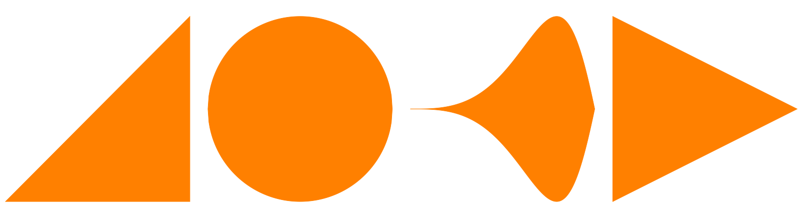
$$\int_0^1 \int_{y-1}^{1-y} y^2\, dx\, dy = 1/6\ .$$

15.9. Let R be the triangle $1 \geq x \geq 0, 0 \leq y \leq x$. What is

$$\int\int_R e^{-x^2}\, dxdy\ ?$$

The left to right integral $\int_0^1 [\int_y^1 e^{-x^2}\, dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions. The bottom to top integral $\int_0^1 [\int_0^x e^{-x^2}\, dy]\, dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\, dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316...\ .$$



15.10. The area of a disc of radius R is $\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\, dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2}\, dx$. Substitute $x = R\sin(u), dx = R\cos(u)du$, to get $\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)}R\cos(u)\, du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u)\, du = R^2\pi$. In the last identity, we have used the **double angle identity** $2\cos^2(x) = 1+\cos(2x)$.

Homework

This homework is due on Tuesday, 7/19/2022.

Problem 15.1: Find the area of the region

$$R = \{(x, y) \mid 0 \leq x \leq 4\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value $\int\int_R f(x, y)\ dxdy/\text{area}(R)$ of $f(x, y) = y$ over that region.

Problem 15.2: a) (4 points) Find the iterated integral

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2 + (y^2/2)}\, dy\, dx\ .$$

b) (4 points) Now compute

$$\int_0^1 \int_0^2 6xy/\sqrt{x^2 + y^2/2}\, dx\, dy\ .$$

c) (2 points) Wouldn't Fubini assure that a) and b) are the same? What change would be needed in b) to make the results agree?

Problem 15.3: Find the volume of the solid lying under the paraboloid $z = 3x^2 + 3y^2$ and above the rectangle $R = [-2, 2] \times [-2, 4] = \{(x, y) \mid -3 \leq x \leq 3, -2 \leq y \leq 4\ \}$.

Problem 15.4: a) Fist evaluate the iterated integral $\int_0^1 \int_x^{2-x} 6(x^2 - y)\ dydx$. Make sure to sketch the corresponding bottom to top region. b) Rewrite the integral as a left to right region and compute the integral again.

Problem 15.5: There is a great way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!
a) (5 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} 44(x^2 + y^2)^{10}\ dxdy\ .$$

You might want to "time travel" one lecture forward, where polar coordinates are known to solve this problem. b) (5 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{3x^7}{\sqrt{x} - x^2}\, dx\, dy\ .$$