MULTIVARIABLE CALCULUS

MATH S-21A

Unit 13: Extrema

Lecture

13.1. In applications we often are led to the task to **maximize** or **minimize** a function f. As in single variable calculus, the strategy is to look for points where the **derivative** is zero. In the interior of an Euclidean domain this is needed for a maximum by the **Fermat principle**. In one dimensions, like for $f(x) = 3x^5 - 5x^3$ we can then use the **second derivative test** to classify the extrema, like **local max** at -1 and the **local min** at 1. [1]

> **Definition:** A point (a, b) in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = [0, 0]$.

13.2. The **Fermat principle** in two dimensions tells:

> If $\nabla f(x, y)$ is not zero, then (x, y) is not a maximum or minimum.

13.3. Proof. Take the directional derivative in the direction $\vec{v} = \nabla f/|\nabla f|$ at $\vec{x} = (x, y)$. Then $D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f| > 0$. This means that $f(\vec{x} + \epsilon\vec{v}) > f(\vec{x})$ and $f(\vec{x} - \epsilon\vec{v}) < f(\vec{x})$ for small ϵ and $\vec{x}$ is neither a maximum nor a minimum. QED

13.4. Note that in the definition, we do **not** include points, where f or its derivative is not defined. For $f(x, y) = |x| + |y|$ we would have to exclude the points on the x and the y axis and study the function there separately. Without stating otherwise, we always assume that a function f can be differentiated arbitrarily often. Points, where the function has no derivatives are just not considered to be part of the domain and need to be studied separately. For the continuous function $f(x, y) = 1/\log(|xy|)$ for example, we would have to look at the points on the coordinate axes as well as the points on the hyperbola $xy = 1$ separately.

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[1] It is custom to abbreviate max for maximum and min for minimum.

13.5. In one dimension, we used the condition $f'(x) = 0, f''(x) > 0$ to get a local minimum and $f'(x) = 0, f''(x) < 0$ to assure a local max. If $f'(x) = 0, f''(x) = 0$, the nature of the critical point is undetermined and could be a max like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat **inflection point** like for $f(x) = x^3$, where we have neither a max nor a min.

Definition: If $f(x, y)$ is a function of two variables with a critical point (a, b), the number $D = f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** of the critical point.

13.6. The discriminant can be remembered better if seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ combining all the second partial derivatives in one entity, a matrix. [2] As of default, we always assume that functions are twice continuously differentiable. Here is the **second derivative test**:

Theorem: Assume (a, b) is a critical point for $f(x, y)$.
If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local min.
If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local max.
If $D < 0$ then (a, b) is a saddle point.

13.7. If $D \neq 0$ at all critical points, the function f is called **Morse**. The Morse condition is nice as for $D = 0$, we need higher derivatives or ad-hoc methods to determine the nature of the critical point.

13.8. To find the max or min of $f(x, y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see in the next unit how to get extrema on the boundary.

13.9. Sometimes, we want to find the overall maximum and not only the local ones.

Definition: A point (a, b) in the plane is called a **global maximum** of $f(x, y)$ if $f(x, y) \leq f(a, b)$ for all (x, y). For example, the point $(0, 0)$ is a global maximum of the function $f(x, y) = 1 - x^2 - y^2$. Similarly, we call (a, b) a **global minimum**, if $f(x, y) \geq f(a, b)$ for all (x, y).

[3]

Examples

13.10. Find the critical points of $f(x, y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x, y) = [4(x^3 - y), 4(y^3 - x)]$. This is $[0, 0]$ at the points $(0, 0), (1, 1), (-1, -1)$. These are the critical points.

13.11. $f(x, y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x, y) = [2x\cos(x^2 + y), \cos(x^2 + y) + 1]$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\dots (0, -\pi), (0, \pi), (0, 3\pi), \dots$. There are infinitely many critical points here.

[2] Matrix comes from Mater = womb of a mother as matrices are the mother of determinants!

[3] We avoid "absolute maximum" as this would suggest to look for the maximum of $|f|$. Compare for example when looking at absolute convergence of series.

13.12. The graph of $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = [2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2)]e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2+y^2 = 1$. This function has a continuum of critical points.

13.13. The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant and related to the gravitational strength. We have $\nabla f = (y, -g\sin(x)) = [(0,0]$ for

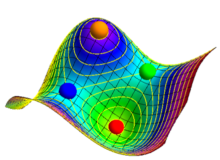
$$(x,y) = \dots, (-\pi, 0), (0,0), (\pi, 0), (2\pi, 0), \dots .$$

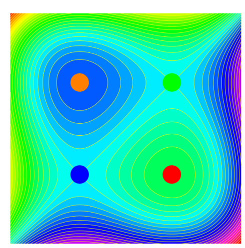
These points are equilibrium points, the angles for which the pendulum is at rest.

13.14. The function $f(x,y) = a\log(y) - by + c\log(x) - dx$ is a function which is invariant by the flow of the **Volterra-Lodka** differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point of f. In the context of differential equations, we say that this is an equilibrium point of the system.

13.15. The function $f(x,y) = |x| + |y|$ is smooth on the first quadrant $\{x > 0, y > 0\}$. It does not have critical points there. The function has a minimum at $(0,0)$ but it is not in the domain, where f and ∇f are defined. We have to look at the points on the coordinate axis separately. For $y = 0$, we see that $x = 0$ is a minimum of $|x|$ For $x = 0$ we see that $y = 0$ is a minimum of $|y|$. Now $(0,0)$ is a minimum of f. This minimum was not detected using derivatives.

13.16. The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x,y) = [x^2 - 1, -y^2 + 1]$. There are 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1,1)$ we have $D = -4$ and so a saddle point,
For $(-1,1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1,-1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1,-1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





13.17. Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$

is a local maximum on the line $y = -1$. Comparing $f(4/3, 0), f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

13.18. Find the global maxima and minima of $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x, 0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

Homework

This homework is due on Tuesday, 7/19/2022.

Problem 13.1: Find all the extrema of the function

$$f(x, y) = xy + x^2 y - xy^2$$

they are maxima, minima or saddle points.

Problem 13.2: Where on the parametrized surface $\vec{r}(u, v) = [1 + u^3, v^2, uv]$ is the temperature $T(x, y, z) = 7 + x + 12y - 24z$ minimal? To find the minimum, minimize the function $f(u, v) = T(\vec{r}(u, v))$. Find all local maxima, local minima or saddle points of f.

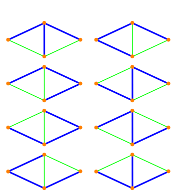
Problem 13.3: Do this problem as it is an old classic and because it will appear again in HW 14. Find and classify all the extrema of the function $f(x, y) = e^{-x^2 - y^2}(x^2 + 2y^2)$.

Problem 13.4: Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 13$e and characterize them. Do you find a global maximum or global minimum among them?

Problem 13.5: Graph theorists are fond of at the **Tutte polynomial** $f(x, y)$ of a network. We work with the Tutte polynomial

$$f(x, y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**. Classify using the second derivative test.



Remark. The polynomial is useful: $xf(1 - x, 0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1, 1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1, 1) = 8$ as you see the 8 possible trees.