MULTIVARIABLE CALCULUS

MATH S-21A

Unit 12: Tangent spaces

Lecture

12.1. The **gradient** $\nabla f(x,y,z) = [a,b,c]$ is the derivative of a scalar function f of many variables. It produces a vector at every point (x,y,z). This vector $[a,b,c]$ is useful for example to compute tangent lines or tangent planes.

Definition: The **gradient** of a function $f(x,y)$ is defined as

$$\nabla f(x,y) = [f_x(x,y), f_y(x,y)] \ .$$

For functions of three variables, define

$$\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)] \ .$$

12.2. The symbol ∇ is spelled "Nabla" and named after an Egyptian or Assyrian harp. Early on, the name "Atled" was suggested. But the textbook of 1901 of Gibbs used Nabla was too persuasive. The following important fact holds in any dimension.

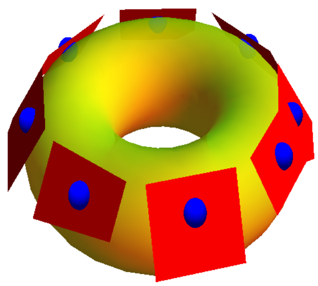
Theorem: **Gradient Theorem:** $\nabla f(x_0,y_0)$ is perpendicular to the level curve $\{(x,y) \mid f(x,y) = c\}$ containing (x_0,y_0). $\nabla f(x_0,y_0,z_0)$ is perpendicular to the level surface $\{(x,y,z) \mid f(x,y,z) = c\}$ containing (x_0,y_0,z_0).

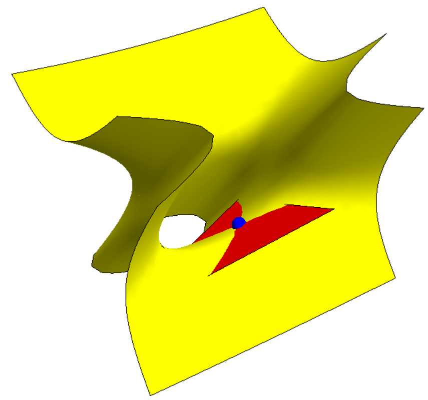
Proof: Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$. QED.

12.3. Because $\vec{n} = \nabla f(p,q) = [a,b]$ is perpendicular to the level curve $f(x,y) = c$ through (p,q), the equation for the tangent line is $ax+by = d$, $a = f_x(p,q)$, $b = f_y(p,q)$, $d = ap + bq$. Compactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus as it gives a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines, without linearization!





Definition: If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

The name "directional derivative" is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x, y)$, then $D_{\vec{v}} h(x, y)$ gives us the slope we drive on.

12.4. If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}\,'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

12.5. For $\vec{v} = [1, 0, 0]$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

12.6. The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|}$ because

$$\nabla f \cdot \vec{v} \leq |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}| \ .$$

Definition: The direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

12.7. If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

12.8. If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}\,'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

12.9. If $\vec{v} = \nabla f/|\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

12.10. The directional derivative has the same properties than any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f+g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + fD_v(g)$.
We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **critical points** and help to understand the function f.

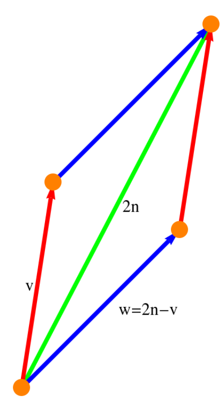
Examples

12.11. Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1, 1, 1)$. **Solution:** $\nabla f(x, y, z) = [6xy, 3x^2, 2z]$. And $\nabla f(1, 1, 1) = [6, 3, 2]$. The plane is $6x + 3y + 2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x + 3y + 2z = 11$.

12.12. **Problem:** reflect the ray $\vec{r}(t) = [1 - t, -t, 1]$ at the surface

$$x^4 + y^2 + z^6 = 6 .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1, -2, 1)$. The velocity vector in that ray is $\vec{v} = [-1, -1, 0]$. The normal vector at this point is $\nabla f(-1, -2, 1) = [-4, -4, 6] = \vec{n}$. The reflected vector is $R(\vec{v} = 2\text{Proj}_{\vec{n}}(\vec{v}) - \vec{v}$. We have $\text{Proj}_{\vec{n}}(\vec{v}) = 8/68[-4, -4, 6]$. Therefore, the reflected ray is $\vec{w} = (4/17)[-4, -4, 6] - [-1, -1, 0]$.



12.13. You are on a trip in a air-ship over Cambridge at $(1, 2)$ and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest? **Solution:** The gradient $\nabla p(x, y) = [2x, 4y]$ at the point $(1, 2)$ is $[2, 8]$. Normalize to get the direction $[1, 4]/\sqrt{17}$.

12.14. The "Dom" is a mountain in Switzerland with an altitude of 4'545 meters. In suitable units on the ground, the height $f(x, y)$ is approximated by the quadratic function $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the south-east direction $\vec{v} = [1, -1]/\sqrt{2}$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = [-2x, -2y]$, so that $[20, -20] \cdot [1, -1]/\sqrt{2} = 40/\sqrt{2}$. This is a place, with a ladder, where you climb $40/\sqrt{2}$ meters up when advancing 1m forward. The rate of change in all directions is zero if and only if $\nabla f(x, y) = [0, 0]$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f/|\nabla f|$ and get $D_{\nabla f} f = |\nabla f|$.

Dom as seen from the Alp Salmenfee in Switzerland. Oliver was finally back there this summer 2022.

12.15. Assume we know $D_v f(1, 1) = 3/\sqrt{5}$ and $D_w f(1, 1) = 5/\sqrt{5}$, where $v = [1, 2]/\sqrt{5}$ and $w = [2, 1]/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f. **Solution:** Let $\nabla f(1, 1) = [a, b]$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.

Homework

This homework is due on Tuesday, 7/12/2022.

Problem 12.1: Find the directional derivative $D_{\vec{v}}f(3,1) = \nabla f(3,1) \cdot \vec{v}$ into the direction $\vec{v} = [3,-4]/5$ for the function $f(x,y) = 2 + x^4 y + y^2 + y$.

Problem 12.2: A surface $x^2 + y^2 - z = 1$ radiates light away. It can be parametrized as $\vec{r}(x,y) = [x, y, x^2 + y^2 - 1]$. Find the parametrization of the wave front $\vec{r}(x,y) + \vec{n}(x,y)$, which is distance 1 from the surface. Here $\vec{n}$ is a unit vector normal to the surface.

Problem 12.3: Assume $f(x,y) = 1 - x^2 + y^2$. Compute the directional derivative $D_{\vec{v}}(x,y)$ at $(0,0)$, where $\vec{v} = [\cos(t), \sin(t)]$ is a unit vector. Now compute

$$D_v D_v f(x,y)$$

at $(0,0)$, for any unit vector. For which values t is this **second directional derivative** positive?

Problem 12.4: The **Kitchen-Rosenberg formula** gives the curvature of a level curve $f(x,y) = c$ as

$$\kappa = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

Use this formula to find the curvature of the ellipse $f(x,y) = x^2 + 2y^2 = 1$ at the point $(1,0)$.

This formula is useful in computer vision. If you want to derive the formula, you can check that the angle

$$g(x,y) = \arctan(f_y/f_x)$$

of the gradient vector has κ as the directional derivative in the direction $\vec{v} = [-f_y, f_x]/\sqrt{f_x^2 + f_y^2}$ tangent to the curve.

Problem 12.5: Using gradient methods is is one of the important paradigms in machine learning. One can find the maximum of a function numerically by moving in the direction of the gradient. This is called the **steepest ascent method**. You start at a point (x_0, y_0) then move in the direction of the gradient for some time c to be at $(x_1, y_1) = (x_0, y_0) + c\nabla f(x_0, y_0)$. Repeat to $(x_2, y_2) = (x_1, y_1) + c\nabla f(x_1, y_1)$ etc. It can be a bit difficult if the function has a flat ridge like in the **Rosenbrock function**

$$f(x,y) = 1 - (1-x)^2 - 100(y - x^2)^2 \ .$$

Plot the contour map of this function on $-0.6 \leq x \leq 1, -0.1 \leq y \leq 1.1$, then and find the directional derivative at $(1/5, 0)$ in the direction $(1,1)/\sqrt{2}$.

Multivariable Calculus

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