MULTIVARIABLE CALCULUS

MATH S-21A

Unit 8: Arc length and Curvature

Lecture

Definition: If $t \in [a,b] \mapsto \vec{r}(t)$ parametrizes a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\, dt$ is called the **arc length** of the curve.

8.1. If $\vec{r}$ is differentiable, then a polygon approximation justifies it. Written out, the formula is $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$.

8.2. Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately see "path independence of arc length":

The arc length is independent of the parameterization of the curve.

Definition: Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$.

Definition: The **curvature** of a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

8.3. The curvature is the length of the acceleration vector if $\vec{r}(t)$ parametrizes the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration. We can measure curvature at a point only if $\vec{r}\,'(t)$ is not zero.

The curvature does not depend on the parametrization.

Proof. Let $s(t)$ be an other parametrization, then by the chain rule $d/dtT(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

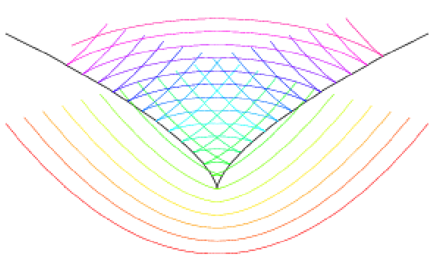
Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

Definition: If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by $\vec{N}$ and $\vec{B}$ is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

8.4. If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because $\vec{B}$ is automatically normal to $\vec{T}$ and $\vec{N}$, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

8.5. Here is an application of curvature: if a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call it the **evolute** of the original curve. To the left a caustic. The picture of John Harvard was obtained by following level curves.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

8.6. We prove this in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.

Examples

The arc length of the **circle** $\vec{r}(t) = [R\cos(t), R\sin(t)]$ is $2\pi R$. The speed $|\vec{r}'(t)|$ is constant and equal to R.

The **helix** $\vec{r}(t) = [\cos(t), \sin(t), t]$ has velocity $\vec{r}\,'(t) = [-\sin(t), \cos(t), 1]$ and constant speed $|\vec{r}\,'(t)| = |[-\sin(t), \cos(t), 1]| = \sqrt{2}$.

What is the arc length of the curve

$$\vec{r}(t) = [\sqrt{2}t, \log(t), t^2/2]$$

for $1 \leq t \leq 2$? Answer: Because $\vec{r}\,'(t) = [\sqrt{2}, 1/t, t]$, we have $\vec{r}'(t) = \sqrt{2 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t \; dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity".

Find the arc length of the curve $\vec{r}(t) = [3t^2, 6t, t^3]$ from $t = 1$ to $t = 3$.

What is the arc length of the curve $\vec{r}(t) = [\cos^3(t), \sin^3(t)]$? Answer: We have $|\vec{r}\,'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi} (3/2)\sin(2t) \; dt = 6$.

Find the arc length of $\vec{r}(t) = [t^2/2, t^3/3]$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2} \; dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2} \; dx = 2\int_0^1 x\sqrt{1+x^2} \; dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

The arc length of an **epicycle** $\vec{r}(t) = [t + \sin(t), \cos(t)]$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r}'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)} \;\; dt$. A **substitution** $t = 2u$ gives $L = \int_0^{\pi} \sqrt{2 + 2\cos(2u)} \;\; 2du = \int_0^{\pi} \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)} \;\; 2du = \int_0^{\pi} \sqrt{4\cos^2(u)} \; 2du = 4\int_0^{\pi} |\cos(u)| \; du = 8$.

Find the arc length of the **catenary** $\vec{r}(t) = [t, \cosh(t)]$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1 \; ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. Solution: We have $|\vec{r}\,'(t)| = \sqrt{1 + \sinh^2(t)} = \cosh(t)$ and $\int_{-1}^1 \cosh(t) \; dt = 2\sinh(1)$.

Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = [\cos(3t), \sin(5t)]$ leads to the arc length integral $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)} \; dt$ which can only be evaluated numerically.

The curve $\vec{r}(t) = [t, f(t)]$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1+f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1+f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1+\cos^2(t)}^3$.

Homework

This homework is due on Tuesday, 7/5/2022.

Problem 8.1: a) Find the arc length of the curve $\vec{r}(t) = [t^2, 2t^3/3, 2]$ from $t = -2$ to $t = 2$.
b) Find the arc length of $\vec{r}(t) = [4t, 4\sin(3t), 4\cos(3t), 2]$ with $0 \leq t \leq \pi$.

Problem 8.2: Find the curvature of $\vec{r}(t) = [e^t \cos(t), e^t \sin(t), t]$ at the point $(1, 0, 0)$.

Problem 8.3: Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = [t^2, t^3, 0]$ for $t = 2$. Do the vectors $\vec{T}(t), \vec{N}(t), \vec{B}(t)$ depend continuously t for all t?

Problem 8.4: Let $\vec{r}(t) = [t, t^2]$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$. It is known also as the **evolute** of the curve.

Problem 8.5: If $\vec{r}(t) = [-\sin(t), \cos(t)]$ is the boundary of a coffee cup and light enters in the direction $[-1, 0]$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.