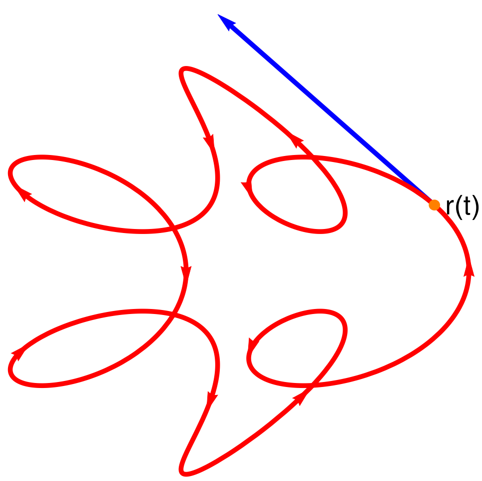
MULTIVARIABLE CALCULUS

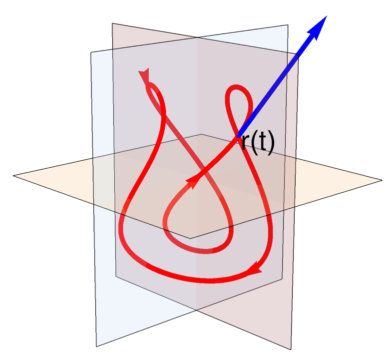
MATH S-21A

Unit 7: Parametrized curves

Lecture

Definition: A **parametrization** of a **planar curve** is a map $\vec{r}(t) = [x(t), y(t)]$ from a **parameter interval** $R = [a, b]$ to the plane $\mathbb{R}^2$. The functions $x(t)$ and $y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. Similarly, the parametrization of a **space curve** is $\vec{r}(t) = [x(t), y(t), z(t)]$. The image of $\vec{r}$ is a **parametrized curve** in space.





7.1. We think of the **parameter** t as **time** and the parametrization as a **drawing process**. The curve is the result what you **see**. For a fixed time t, we have a vector $[x(t), y(t), z(t)]$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains **more information** about the curve then the curve itself. It tells for example how fast the curve was traced.

7.2. Curves can describe the paths of particles, celestial bodies, or other quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time. Here are some more places, where curves appear:

Knots	are closed curves in space.
Molecules	DNA, RNA or proteins.
Graphics:	grid curves produce a mesh of curves.
Typography:	fonts represented by Bézier curves.
Relativity:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

Definition: Any vector parallel to the velocity $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

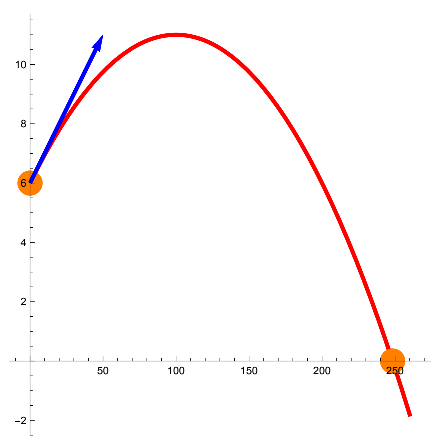
7.3. You know from single variable, the **addition rule** $(f+g)' = f' + g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ as well as the **chain rule** $(f(g))' = f'(g)g'$. They generalize to vector-valued functions.

$(\vec{v}+\vec{w})' = \vec{v}' + \vec{w}'$, $(c\vec{v})' = c\vec{v}'$, $(\vec{v}\cdot\vec{w})' = \vec{v}'\cdot\vec{w} + \vec{v}\cdot\vec{w}'$, $(\vec{v}\times\vec{w})' = \vec{v}'\times\vec{w} + \vec{v}\times\vec{w}'$, $(\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t)$.

7.4. The Differentiation of curves can be reversed using the **fundamental theorem of calculus**. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\,ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ at all times as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\,ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\,ds$.

The **free fall** is the case when acceleration is a constant vector. The direction of the constant force defines what is “down”. If $\vec{r}''(t) = [0,0,-10]$, $\vec{r}'(0) = [0,1000,2]$, $\vec{r}(0) = [0,0,h]$, then $\vec{r}(t) = [0,1000t, h+2t-10t^2/2]$.

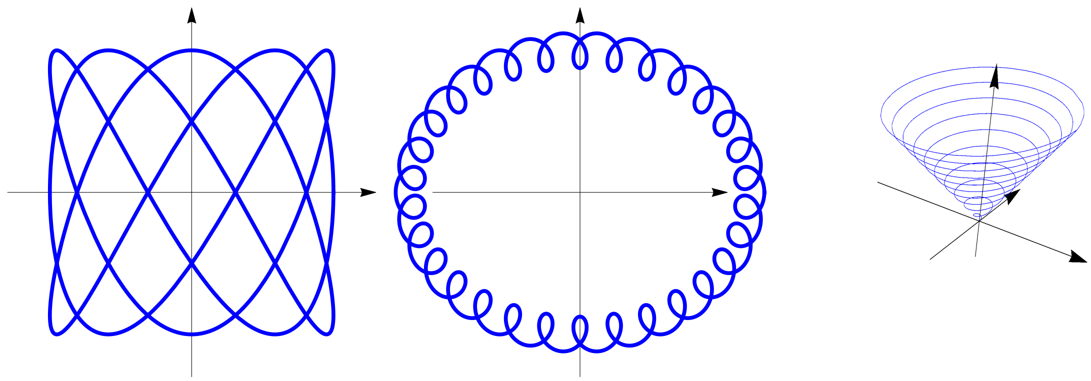


If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2$.

Examples

7.5. Examples:
1) The parametrization $\vec{r}(t) = [1+2\cos(t), 3+5\sin(t)]$ is the ellipse $(x-1)^2/4 + (y-3)^2/25 = 1$. The parametrization $\vec{r}(t) = [\cos(3t), \sin(5t)]$ is an example of a **Lissajous**

curve.
2) If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = [t, f(t)]$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola. 3) With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\vec{r}(t) = [t\cos(t), t\sin(t), t]$ which traces a spiral on a cone $x^2 + y^2 = z^2$. 4) For $x(t) = 2t\cos(2t), y(t) = 2t\sin(2t), z(t) = 2t$ traces the same curve but twice as fast.
5) If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = [a + t(u - a), b + t(v - b), c + t(w - c)]$ with $t \in [0, 1]$ is a **line segment** from P to Q. Example: $\vec{r}(t) = [1 + t, 1 - t, 2 + 3t]$ connects $P = (1, 1, 2)$ with $Q = (2, 0, 5)$.
6) For $\vec{r}(t) = [\cos(t), \sin(2t), 0]$ we get a figure 8 curve.



The computation is done coordinate wise:

$$\begin{array}{lll} \text{Position} & \vec{r}\,(t) & = [\cos(3t), \sin(2t), 2\sin(t)] \\ \text{Velocity} & \vec{r}\,'(t) & = [-3\sin(3t), 2\cos(2t), 2\cos(t)] \\ \text{Acceleration} & \vec{r}\,''(t) & = [-9\cos(3t), -4\sin(2t), -2\sin(t)] \\ \text{Jerk} & \vec{r}\,'''(t) & = [27\sin(3t), 8\cos(2t), -2\cos(t)] \end{array}$$

7.6. Lets look at some examples of velocities and accelerations:

Example	**Velocity**
Hair growth:	0.000000005 m/s
Garden Snail	0.013 m/s
Signals in nerves:	50 m/s
Sound in air:	340 m/s
Speed of bullet:	1'200 m/s
Earth in solar system	30'000 m/s
Sun in galaxy:	200'000 m/s
Light in vacuum:	299'792'458 m/s

Example	**Acceleration**
Train:	0.1-0.3 m/s^2
Sprinter (100 m Dash):	3 m/s^2
Car:	3-8 m/s^2
Free fall:	1G = 9.81 m/s^2
Space X Starship:	4G m/s^2
Combat plane F35A:	9G m/s^2
Ejection from F35A:	14G m/s^2.
Electron in vacuum:	10^{15} m/s^2

Homework

This homework is due on Tuesday, 7/5/2022.

Problem 7.1: Sketch the plane curve
$$\vec{r}(t) = [x(t), y(t)] = [\cos(t) + \sin(5t), \sin(t) + \cos(5t)] \ ,$$
for $t \in [0, 2\pi]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as the acceleration $\vec{r}\,''(t)$ at $t = 0$. When plotting the curve you see a flower shaped curve. How many petals are there?

Problem 7.2: Oliver's cellphone app measures the acceleration
$$\vec{r}\,''(t) = [-\cos(t), -81\sin(9t), 81\cos(9t) - \sin(t)]$$
on a Phoenix coaster in a amusement park. Assume $r(0) = [1, 0, -1]$ and $r'(0) = [0, 9, 1]$. What is its position $\vec{r}(\pi/2)$?

[1]

Problem 7.3: a) Two particles travel along space curves. The first is
$$\vec{r}_1(t) = [t, t^2, t^3] \ .$$
The second is
$$\vec{r}_2(t) = [1 + 2t, 1 + 6t, 1 + 14t] \ .$$
Do the particles collide? Do the particle paths intersect?
b) If $\vec{r}(t) = [\cos(t), 2\sin(t), 4t]$, find $\vec{r}\,'(0)$ and $\vec{r}\,''(0)$. Then compute $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$. We will later call this the curvature.

Problem 7.4: Find the parametrization $\vec{r}(t) = [x(t), y(t), z(t)]$ of the curve obtained by intersecting the elliptical cylinder $x^2/16 + y^2/25 = 1$ with the surface $z = x^2 y$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

Problem 7.5: Consider the curve
$$\vec{r}(t) = [x(t), y(t), z(t)] = [t^2, 1 + t, 1 + t^3] \ .$$
Check that it passes through the point $(1, 0, 0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

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[1]Oliver got really sick on that ride: https://www.youtube.com/watch?v=Hr8h-WmgX0Q