**MULTIVARIABLE CALCULUS**

MATH S-21A

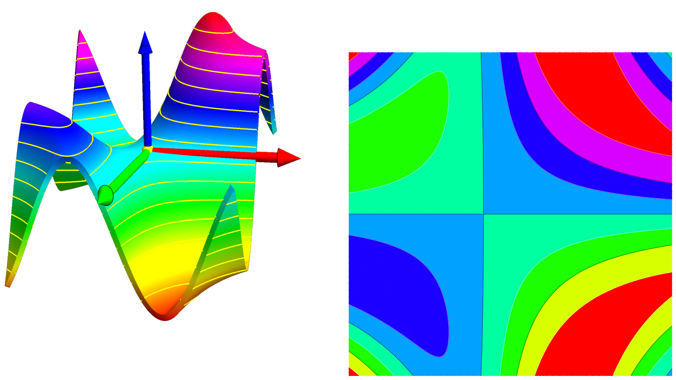
# Unit 5: Functions

Lecture

**5.1.** Functions are a way to define geometric objects or to assign properties to a point of a geometric object. The shape of a mountain for example can be described by a function which assigns to every point $(x, y)$ the height.

> **Definition:** A **function of two variables** $f(x, y)$ is a rule which assigns to two numbers $x, y$ a third number $f(x, y)$.

The function $f(x, y) = x^3y - 2y^2$ for example assigns to $(2, 3)$ the number $24 - 18 = 6$.



**5.2.** In general, a function is assumed to be defined for all points $(x, y)$ in $\mathbb{R}^2$. An example is $f(x, y) = |x|^7 + e^{\sin(xy)}$. Sometimes however it is required to restrict the function to a **domain** $D$ in the plane. For example, if $f(x, y) = \log |y| + \sqrt{x}$, then $(x, y)$ is only defined for $x > 0$ and for $y \neq 0$. The **range** of a function $f$ is the set of values which the function $f$ reaches. The function $f(x, y) = 3+x^2/(1+x^2)$ for example takes all values $3 \leq z < 4$. While $z = 3$ is reached for $x = y = 0$, and all values $z < 4$ can be reached, the value $z = 4$ is not attained.

**5.3.**

> **Definition:** The set $\{(x, y, f(x, y)) \mid (x, y) \in D \} \subset \mathbb{R}^3$ is the **graph** of $f$.

Graphs are **surfaces** which allow to visualize the function. We should not mix up the graph of a function with the function itself. The function is a **rule** which assigns to $(x, y)$ a third number, the graph is a **geometric object** in three dimensional space.

The modern notion of function only started to appear at the beginning of the 19th century. That the function $f$ and the graph of $f$ are different things matters in computer science for example. They are implemented as different data structures. In some cases, we know $f$ but we do not understand the graph. An example is the **zeta function** $f(x,y) = |\zeta(x+iy)|$ for which we know the function very well, can evaluate it at every point but where we do not know the graph, especially, where its zeros are.

**5.4.** Here are some examples:

| Example $f(x,y)$ | domain $D$ of $f$ | range $= f(D)$ of $f$ |
|---|---|---|
| $\sqrt{9-x^2-y^2}$ | closed disc $x^2+y^2 \leq 9$ | $[0,3]$ |
| $-\log(1-x^2-y^2)$ | open unit disc $x^2+y^2 < 1$ | $(0,\infty)$ |
| $f(x,y) = x^2+y^3-xy+\cos(xy)$ | plane $R^2$ | the real line |
| $\sqrt{4-x^2-2y^2}$ | $x^2+2y^2 \leq 4$ | $[0,2]$ |
| $1/(x^2+y^2-1)$ | all except unit circle | $R \setminus (-1,0]$ |
| $1/(x^2+y^2)^2$ | all except origin | positive real axis |

**5.5.**

**Definition:** The set $\{(x,y) \mid f(x,y) = c = \text{const}\}$ is called a **contour curve** or **level curve** of $f$. A collection of contour curves is a **contour map**.

For example, for $f(x,y) = 4x^2 + 3y^2$, the level curves $f = c$ are **ellipses** if $c > 0$. Drawing several contour curves $\{f(x,y) = c\}$ simultaneously produces a **contour map** of the function $f$.

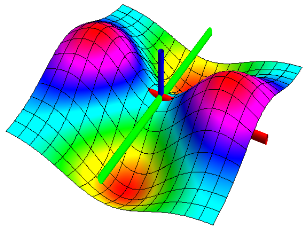
**5.6.** Level curves allow to visualize and analyze functions $f(x,y)$ without leaving the two dimensional space. The picture below for example shows the level curves of the function $\sin(xy) - \sin(x^2+y)$. Contour curves are everywhere: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

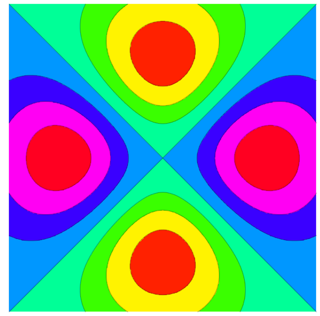
## Examples

**5.7.** For $f(x,y) = x^2 - y^2$, the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

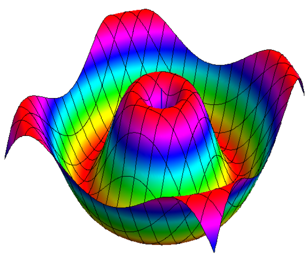
**5.8.** The function $f(x,y) = 1 - 2x^2 - y^2$ has contour curves $f(x,y) = 1 - 2x^2 - y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.

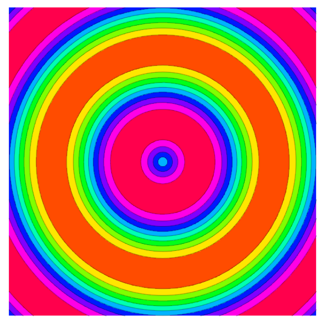
**5.9.** For the function $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2} = c$. We can draw the traces curve $(x,0,f(x,0))$ or $(0,y,f(0,y))$ or then use a computer:





**5.10.** The surface $z = f(x, y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.





**5.11.** In applications, discontinuous functions can occur. The temperature of water in relation to pressure and volume is an example. One experiences then **phase transitions**, places where the function value can jumps. Mathematicians study singularities in a mathematical field called "catastrophe theory".

**5.12.**

**Definition:** A function $f(x, y)$ is called **continuous** at $(a, b)$ if there is a finite value $f(a, b)$ with $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. This means that for any sequence $(x_n, y_n)$ converging to $(a, b)$, also $f(x_n, y_n) \to f(a, b)$. A function is **continuous** in $G \subset \mathbb{R}^2$ if it is continuous at every point $(a, b)$ of $G$.

**5.13.** Continuity means that if $(x, y)$ is close to $(a, b)$, then $f(x, y)$ must be close to $f(a, b)$. Continuity for functions of more than two variables is defined in the same way. The bad news is that continuity is not always easy to check. The good news is that in general we do not have to worry about continuity. Lets look at some examples:

**5.14. Example:** For $f(x, y) = (xy)/(x^2 + y^2)$, we have

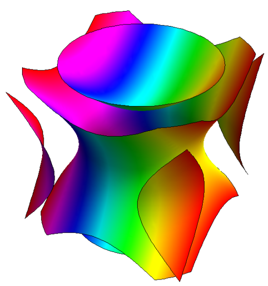
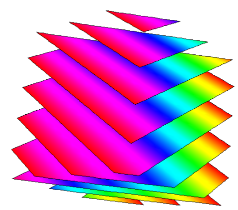
$$\lim_{x\to 0} f(x, x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

and $\lim_{x\to 0} f(0, x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous at $(0, 0)$.

**5.15. Example:** The function $f(x, y) = (x^2 y)/(x^2+y^2)$ is better described using polar coordinates: $f(r, \theta) = r^3 \cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r, \theta) \to 0$ uniformly in $\theta$ if $r \to 0$. The function is continuous as we can extend it and extend the value to $f(0, 0) = 0$. It is custom in mathematics to consider the above function **to be continuous**. The reason is that there is a **unique way** to give a function value at the undefined point.

**5.16.** A simpler example: the function $f(x,y) = (x^2 - y^2)/(x+y)$ is continuous everywhere. Yes, the function is not defined a priori at $x+y=0$ but as it is outside this line equal to $f(x,y) = x - y$, there is a unique continuation to the entire plane and this continuation is $x - y$.

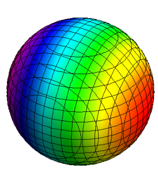
**5.17.** A function of three variables $g(x,y,z)$ assigns to three variables $x, y, z$ a real number $g(x,y,z)$. The function $f(x,y,z) = x^2 + y - z$ for example satisfies $f(3,2,1) = 10$. We can visualize a function by **contour surfaces** $g(x,y,z) = c$, where $c$ is constant. It is an **implicit description** of the surface. The contour surface of $g(x,y,z) = x^2 + y^2 + z^2 = c$ is a sphere if $c > 0$. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.


**5.18.** The function $g(x,y,z) = 2 + \sin(xyz)$ could define a temperature distribution in space. We can no more draw the graph of $g$ because that would be an object in 4 dimensions. We can however draw **level surfaces** like $g(x,y,z) = 0$ or $g(x,y,z) = 1$.

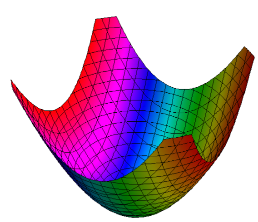
**5.19.** The level surfaces of $g(x,y,z) = x^2 + y^2 + z^2$ are spheres. The level surfaces of $g(x,y,z) = 2x^2 + y^2 + 3z^2$ are ellipsoids. The equation $ax + by + cz = d$ is a plane. With $\vec{n} = [a,b,c]$ and $\vec{x} = [x,y,z]$, we can rewrite the equation $\vec{n} \cdot \vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n} \cdot \vec{x}_0 = d$. so that $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$. This means that every vector $\vec{x} - \vec{x}_0$ in the plane is orthogonal to $\vec{n}$. For $f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m$ the surface $f(x,y,z) = 0$ is called a **quadric**.

Sphere



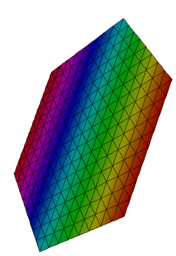
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

Paraboloid



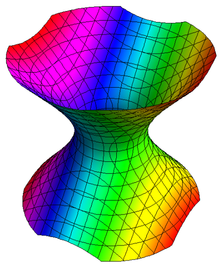
$(x-a)^2 + (y-b)^2 - c = z$

Plane



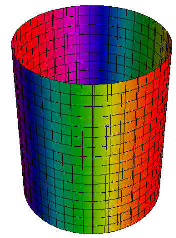
$ax + by + cz = d$

One sheeted hyperboloid



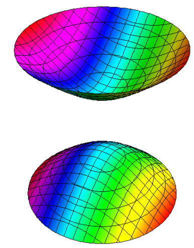
$(x-a)^2+(y-b)^2-(z-c)^2 = r^2$

Cylinder



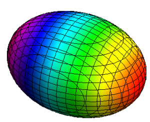
$(x-a)^2+(y-b)^2 = r^2$

Two sheeted hyperboloid



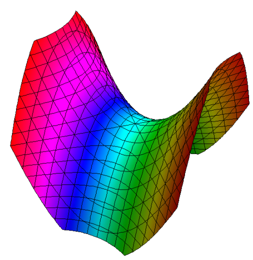
$(x-a)^2+(y-b)^2-(z-c)^2 = -r^2$

Ellipsoid



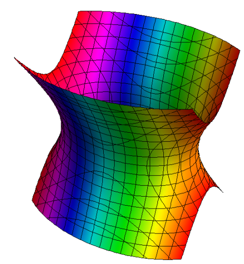
$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Hyperbolic paraboloid



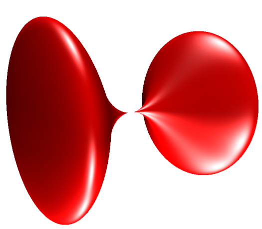
$x^2 - y^2 + z = 1$

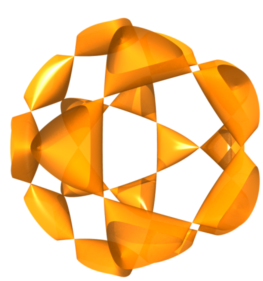
Elliptic hyperboloid

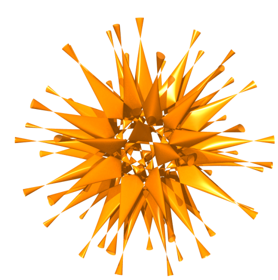


$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$

Higher order polynomial surfaces can be intriguingly beautiful and are sometimes difficult to describe. If $f$ is a polynomial in several variables and $f(x,x,x)$ is a polynomial of degree $d$, then $f$ is called a **degree** $d$ **polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.







# Homework

This homework is due on Tuesday, 7/5/2022.

**Problem 5.1:** a) Plot the contour plot and graph of the function $f(x, y) = \sin(x^2 + y^2)/(x^2 + y^2)$.
b) Plot both the graph and the contour plot of the function $f(x, y) = \sin(x^2) + \sin(y^2)$ on the region $-\pi \leq x \leq \pi, -\pi \leq y \leq \pi$. Try a) without a computer, for b), you might want to use Wolfram alpha or Mathematica or a graphing calculator to get your picture.

**Problem 5.2:** a) Determine the domain and range of the **logarithmic mean**

$$f(x, y) = \frac{(y - x)}{\log(y) - \log(x)}$$

where log the natural logarithm.
b) The function is not defined at $x = y$ but one can define $f(x, y)$ on the diagonal $x = y$. Use Hôpital, show that the limit $\lim_{x\to 2} f(x, 2)$ exists.
c) The function is also not defined at first if $x = 0$ or $y = 0$. Show that the limit $\lim_{x\to 0} f(x, 2)$ exists.

**Problem 5.3:** a) Use the computer to draw the level surface $x^2 - y^2 + z^2 - x^4y^4z^4 - x^2y^2z^2 = 0$ with $x, y, z$ all in $[-2, 2]$
b) Do the same for the contour $((x^2 + y^2)^2 - x^2 - y^2)^2 + z^2 = 0.02$ with with $x, y, z$ all in $[-1.1, 1.1]$

**Problem 5.4:** a) Draw the Taxi-metric hyperboloid $|x| + |y| - |z| = 1$.
b) Draw the Taxi-metric hyperboloid $|x| + |y| - |z| = -1$.
c) Draw the Taxi-metric ellipsoid $|x| + |y| + 2|z| = 5$.
d) Draw the Taxi-metric elliptic parabolid $z = |x| + |y|$.
e) Draw the Taxi-metric hyperboloic paraboloid $z = |x| - |y|$.

**Problem 5.5:** a) Verify that the line $\vec{r}(t) = [1 + t, 1 - t, t]$ is part of the one sheeted hyperboloid $x^2 + y^2 - 2z^2 = 2$.
b) Verify that the line $\vec{r}(t) = [1, 3, 2] + t[1, 2, 1]$ is part of the hyperbolic paraboloid $z^2 - x^2 - y = 0$.
c) As also the line $\vec{r}(s) = [1 - s, 1 + s, s]$ is part of the same hyperboloid, what is the intersection of the hyperboloid with the plane $\vec{r}(t, s) = [1 + t - s, 1 - t + s, t + s]$?

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