MULTIVARIABLE CALCULUS

MATH S-21A

Unit 1: Geometry and Distance

Lecture

1.1. A point on the **real line** $\mathbb{R}$ is determined by a single real coordinate x. **Zero** 0 divides the **positive axis** from the **negative axis**. A point $P = (x, y)$ in the **plane** $\mathbb{R}^2$ is fixed by two **coordinates** x and y. In **space** $\mathbb{R}^3$, locating a point needs three coordinates $P = (x, y, z)$. The third coordinate z is usually interpreted as **height**, the distance from the xy-plane. The signs define four **quadrants** in $\mathbb{R}^2$ or eight **octants** in $\mathbb{R}^3$. These regions all intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are bound by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

1.2. In $\mathbb{R}^2$, we usually orient the x-axis to point "east" and the y-axis point "north". In $\mathbb{R}^3$, a common view is to see the xy-plane as the "ground" and to imagine the z-coordinate axis pointing "up". A **photographic coordinate system** appears in computer graphics or photography, where the xy-plane is the **retina** or film plate and the z-coordinate measures the distance towards the viewer.

1.3. The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as

Definition: $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

1.4. The points $A = (1, 1, 0), B = (1, 0, 1), C = (0, 1, 1)$ define an **equilateral triangle** in space because all distances $d(A, B), d(B, C)$ and $d(A, C)$ are equal. The points $A = (275, 0, 0), B = (0, 252, 0), C = (0, 0, 240)$ define a triangle, in which all sides have integer length. You will encounter this triangle in a homework.

1.5. The distance formula is a **definition**, not a result. It is **motivated** by the **theorem of Pythagoras**. We will **prove** the Pythagorean theorem later. This distance is defined in any dimension. In the plane for example, the distance of the point (x, y) to (a, b) is $\sqrt{(x - a)^2 + (y - b)^2}$. When working in $\mathbb{R}^2$, we do not think of it as part of $\mathbb{R}^3$. Coordinates work in arbitrary dimensions. A collection of n data points defines a vector in $\mathbb{R}^n$. The Euclidean space $\mathbb{R}^n$ appears in **data science** as n real data points can be seen as a vector. The Euclidean distance between two data points $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n)$ is then $d(x, a)^2 = \sum_{k=1}^{n} (x_k - a_k)^2$. The sum of the squares appears in statistics, like in **least square problems**.

1.6. Points, **curves**, **surfaces** and **solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a **circle**, an example of a surface is a **sphere**, an example of a solid is the **ball**, the region enclosed by a **sphere**.

Definition: A **circle** of radius $r \geq 0$ centered at $P = (a, b)$ is the collection of points in $\mathbb{R}^2$ which have distance r from P. A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in $\mathbb{R}^3$ which have distance $\rho \geq 0$ from P. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

1.7. Completing the square for an equation $x^2+bx+c = 0$ means adding $(b/2)^2-c$ on both sides to get $(x+b/2)^2 = (b/2)^2-c$. Solving for x gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$. This is the **quadratic equation**. It is good to know this equation because you do not want to waste your creative power having to re-derive this. If you still should forget, the completion idea allows you to re-derive it.

Examples

1.8. $P = (-1, -3)$ is in the third quadrant of the plane and $P = (2, 4, 3)$ is in the positive octant of space. The point $(0, 0, -8)$ is located on the negative z axis. The point $P = (4, 5, -3)$ is below the xy-plane. Can you spot the point Q on the xy-plane which is closest to P?

1.9. Problem: Find the midpoint M on the line segment connecting $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The point we are looking for is the average $M = (P + Q)/2$. The distances are $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$, $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. and $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

1.10. The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after a **completion of the square** $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x + 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(-5/2, 1, 0)$ and **radius** $5/2$.

1.11. An other distance $d(P, Q) = |x - a| + |y - b|$ in the plane $\mathbb{R}^2$ is also called the **taxi metric** or **Manhattan distance**. **Problem**: draw a circle of radius 2 in this metric. More challenging: draw an ellipse in this plane: the set of points whose sum of the distances from $(-2, 0)$ and $(2, 0)$ is equal to 6.

1.12. Draw the unit circle of the **quartic distance** $d(x, y) = (x-a)^4 + (y-b)^4$. More generally, for any $p > 1$, we get a distance $d(x, y) = (x-a)^p + (y-b)^p$. For $p = 1$, it is the **taxi metric**, for $t = 2$ it is the **Euclidean metric**, for $t \to \infty$ it goes to the distance $\max(|x-a|, |y-b|)$ which is the l^∞ metric. **Problem**: is $d(P, Q) = \sqrt{|x - a|} + \sqrt{|y - b|}$ a distance? **Answer**: no, while it satisfies $d(P, Q) = d(Q, P)$ and is zero if and only if $P = Q$, it does not satisfy the **triangle inequality** $d(A, B) + d(B, C) \geq d(A, C)$. We call a space (X, d) for which d is a distance formula satisfying $d(P, Q) = d(Q, P)$, $d(P, Q) = 0 \leftrightarrow P = Q$ and $d(A, B) + d(B, C) \geq d(A, C)$ a **metric space**.

1.13. Problem: Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1, 0)$ and $B = (-1, 0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a, 0)$ satisfies this equation, $d(P, A) + d(P, B) = (a - 1) + (a + 1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0, b)$ satisfying it gives $d(Q, A) + d(P, B) = 2\sqrt{b^2 + 1} = 3$ or $b = \sqrt{5}/2$.

1.14. In an appendix to "La Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. But ideas do not grow in a vacuum; Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. One of the first to explore higher dimensional Euclidean space was Ludwig Schläfli. [1]

1.15. The method of completion of squares is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [2]

1.16. The Euclidean geometry described is only one of many geometries. One can work with more general **metric spaces**. An important class of metric spaces are studied in Riemannian geometry, where the distance between two points can become dependent on where we are. Space becomes curved. This is the frame work of general relativity. Formally, this can happen by changing the coefficients E, G of the metric $d(P, Q)^2 = E(x - a)^2 + G(y - b)^2$. One a sphere, where $x = \theta \in [0, 2\pi]$ is longitude and $y = \phi \in [0, \pi]$ is latitude, one would take $E = \sin^2(y), G = 1$. Two points on the arctic circle with fixed longitude have shorter distance than two points on the equator with the same fixed longitudes. It is important to think now of the surface of the sphere as a space itself, without its embedding in the ambient space. This space is curved. Our four dimensional space-time universe is curved depending on the matter distribution.

[1] An entertaining read is "Descartes secret notebook" by Amir Aczel which deals with an other discovery of Descartes.

[2] The book "The mathematics of Egypt, Mesopotamia, China, India and Islam, by Ed Victor Katz, page 542 contains translations of some of this work.

Homework

This homework is due on Tuesday, 6/28/2022.

Problem 1.1: a) Verify that the three points $P = (2,4,0), Q = (4,0,2), R = (6,2,-2)$ define the corners of an equilateral triangle.
b) Find the smallest sphere which passes through all three points.

Problem 1.2: For each of the following objects in $\mathbb{R}^3$ draw it and describe it with a word: a) $A = |x| + |y| + |z| \leq 1$. b) $B = x^2 + y^2 < 1$, c) $C = A \cap B$ A intersected with B, d) $D = A \setminus B$ A without B.
e) With $d((x,y),(a,b)) = |x-a| + |y-b|$, $A = (-1,0), B = (0,1)$, draw $E = \{X = (x,y), d(X,A) + d(X,B) \leq 8\}$.

Problem 1.3: Verify that the radius of the inscribed circle in a $3:4:5$ triangle is 1.

Problem 1.4: The figure shows two rectangles of area 64 and 65 made up of matching pieces. What is going on?

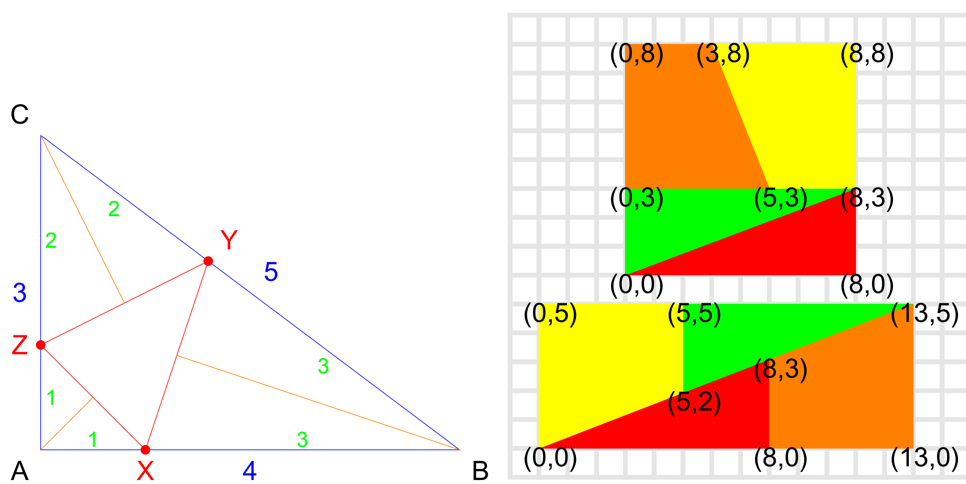

FIGURE 1. The 3-4-5 triangle and the Curry missing square paradox.

Problem 1.5: You play billiard in the table $\{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq 8\}$. a) Hit the ball at $(3,2)$ to reach the hole $(4,8)$ bouncing 3 times at the left wall and three times at the right wall and no other walls. Find the length of the shot.
b) Hit from $(3,2)$ to reach the hole $(4,0)$ after hitting twice the left and twice the right wall as well as the top wall $y = 8$ once. What is the length of the trajectory?