The Unreasonable Effectiveness of Calculus

Calculus is a theory of **change** and **accumulation**. It leads to **derivatives** and **integrals**. These ideas also work in the discrete. Already Gottfried Leibniz illustrated the calculus ideas using discrete notions. We illustrate this here with a **dictionary**, where 24 topics of our summer school course are compared to notions in **discrete calculus**. The later is part of combinatorics. Finite geometric frame-works are **graph theory** or the **simplicial complexes**. The later is a finite collection of finite non-empty sets closed under taking non-empty subsets. If the sets are the nodes and two sets are connected by edges if one is contained in the other, one obtains a graph. One can therefore work with graphs rather than with sets of sets. We can think about a graph as **a space-time geometry** or a **"number" in a larger arithmetic** where one computes with geometries rather than with integers or a **particle configuration in physics**. There have been various attempts to combine the **quantum world** with the **geometry of gravity**. Examples of finitist approaches are **causal sets**, **causal dynamical triangulation** or **loop quantum gravity**. The question has inspired pop culture like in Ant-man (2015) or Interstellar (2014). Which mathematical structures will matter in a **quantum gravity**? It is likely that any future language describing fundamental processes in nature will be a **calculus flavor**. Whatever this theory will be, multi-variable calculus will remain a prototype structure and remain relevant. Like the **Newtonian theory** of planetary motion which has been super-seeded fundamentally by **quantum mechanics** or **relativity**, the path of a space crafts is still computed using classical calculus, maybe using tiny corrections due to special and general relativity. It is important to note that a successful theory will have to be able to do things well in both small as well as large scales: it has to describe the features of atoms and molecules to do chemistry, predict the future of a planetary motion for millions of years or describe correctly large scale physics like black hole mergers. It will also have to say things in situations where current models do not even suggest anything like what happens with the singularity at the end of a black hole evaporation process. This question posed by the physicist Gerard 't Hooft involves both gravity and quantum mechanics. And both of these pillars use the language of multivariable calculus, the topic you learn this summer.

Chapter 1. Geometry and Space

3-manifold	a graph for which every unit sphere is a 2-sphere
2-manifold	a graph for which every unit sphere is a 1-sphere
distance	length of shortest path between two vertices
2-sphere	a 2-manifold which can be collapsed if a point is taken

Chapter 2: Curves and Surfaces

parametrization	collection of adjacent edges or triangles
contour	cliques on which $f - c$ changes sign
arc length	number of edges
surface area	number of triangles

Chapter 3: Linearization and gradient

gradient	exterior derivative d, a function on edges
curl	exterior derivative d, a function on triangles
div	exterior derivative d, a function on tetrahedra
Laplacian	$L = d^*d$

Chapter 4: Extrema and Double integrals

critical point	point a where $\{f(x) < f(a)\}$ is not contractible
discriminant	the index of a critical point
double integral	function over triangles
surface area	number of triangles

Chapter 5: Integrals of fields

scalar integral	sum up 0-form on points (0 manifold)
line integral	sum up 1-form along path (1 manifold)
flux integral	sum of 2-form over surface (2 manifold)
triple integral	sum of 3-form over solid (3 manifold)

Chapter 6: Integral theorems

FTLI	defined for a curve a collection of edges
Green	defined on a surface, a collection of triangles
Stokes	surface with boundary in 3-space, defined for collection of triangles
Gauss	region with boundary in 3-space, defined for a collection of tetrahedra

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