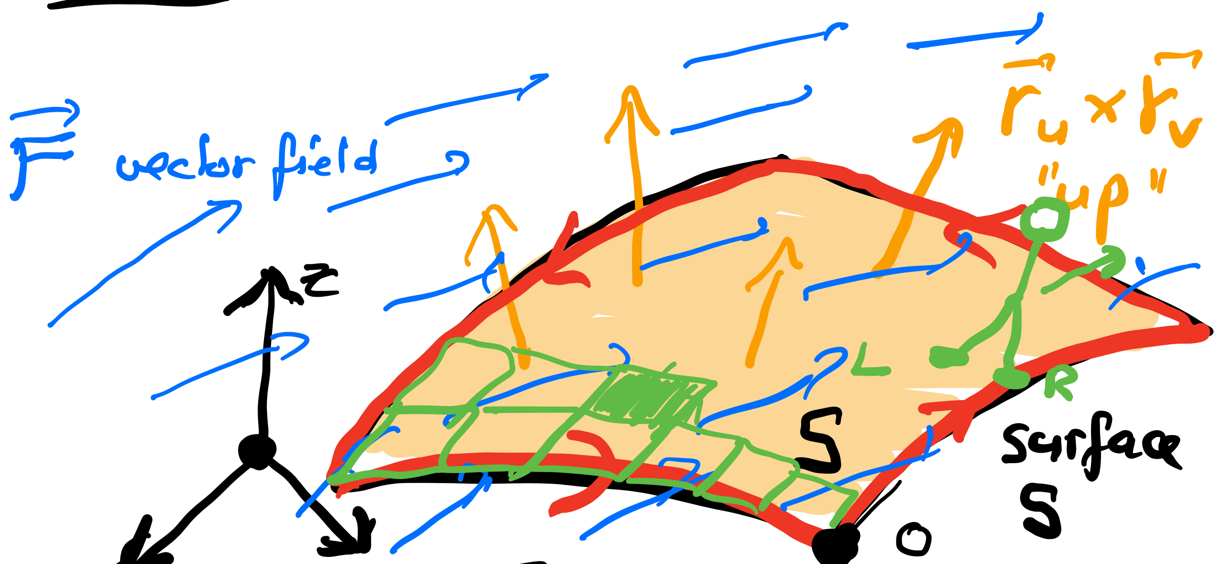
Unit 23

Stokes theorem

today, we live in $\mathbb{R}^3$

1) Theorem

Theorem Stokes Theorem: $$\boxed{\iint_S \text{curl}(\vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}}$$



C boundary curve oriented, so that S is to your left.

left foot on surface, right foot on boundary, head is up.

Lets write this out in detail, and see the meaning and examples.

If $\vec r(u,v) : R \to S$ is the parametrization of S and

$\vec r(t) : [0, 2\pi] \to \mathbb{R}^3$ is the boundary curve.

closed curve

dot prod

flux integral

cross product

$d\vec S$

$d\vec r$

$$\iint_R \text{curl}(\vec F(\vec r(u,v)))\cdot \vec r_u \times \vec r_v \, du\,dv = \int_0^{2\pi} \vec F(\vec r(t))\cdot \vec r'(t)\,dt$$

dA

curl

double integral

dot product

curve

velocity

Line integral

Parametrized Surface

triple scalar prod

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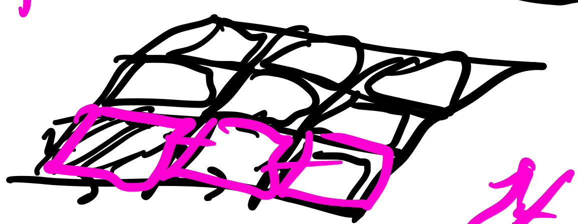
why is this true



Compare: $\iint |\vec r_u \times \vec r_v| \, du\,dv$ was the surface area

$\to$ HW

$dx\,dy = dA$

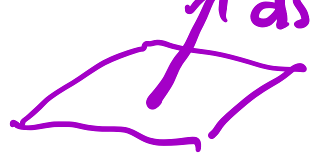
$\vec r_u \times \vec r_v \, du\,dv = d\vec S$

$|\vec r_u \times \vec r_v| \, du\,dv = dS$

$dx\,dy\,dz = dV$

$\vec r'(t)\,dt = d\vec r$

$d\vec S$



notation:

$\iint_S dS =$ surface area

(2) Example

$\partial_x f = \frac{d}{dx} f = f_x$

$S: z = 4 - x^2 - y^2, \; z \geq 0$

oriented upwards.

$$\vec{F} = [-2y, 3x, z^4]$$

Task: compute both sides of Stokes theorem

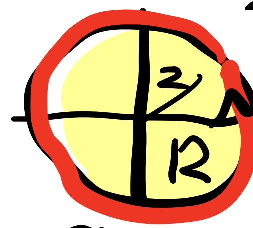
First: LHS: $\iint_S \text{curl}(\vec{F}) \, d\vec{S}$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 1 \\ 0 \\ -2u \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix} = \begin{bmatrix} 2u \\ 2v \\ 1 \end{bmatrix}$$

orient ok

$$\vec{F}(u,v) = [u, v, 4-u^2-v^2]$$

$$R: \quad u^2+v^2 \le 4$$



$$\text{curl}(\vec{F}) = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} -2y \\ 3x \\ z^4 \end{bmatrix}$$

$$= \nabla \times \vec{F}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\iint_R \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2u \\ 2v \\ 1 \end{bmatrix} du\, dv$$

$$= \iint_R 5 \, du\, dv = 5 \cdot \text{Area}(R)$$

$$= 5 \cdot 4\pi = \boxed{20\pi}$$

This is the flux of the Curl of $\vec{F}$.

Second RHS: $\int_C F \cdot dr$

$$\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\vec{F}(\vec{r}(t)) = \begin{bmatrix} -4\sin t \\ 6\cos t \\ 0 \end{bmatrix}$$

$$\vec{r}'(t) = \begin{bmatrix} -2\sin t \\ 2\cos t \\ 0 \end{bmatrix}$$

$$\int_0^{2\pi} \begin{bmatrix} -4\sin t \\ 6\cos t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2\sin t \\ 2\cos t \\ 0 \end{bmatrix} dt$$

$$= \int_0^{2\pi} 8\sin^2 t + 12\cos^2 t \, dt$$

Double angle

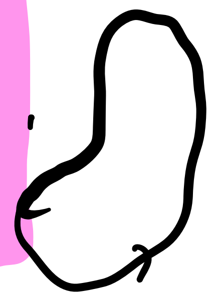
$$= (8+12)\pi = \boxed{20\pi}$$

3 Remarks

closed

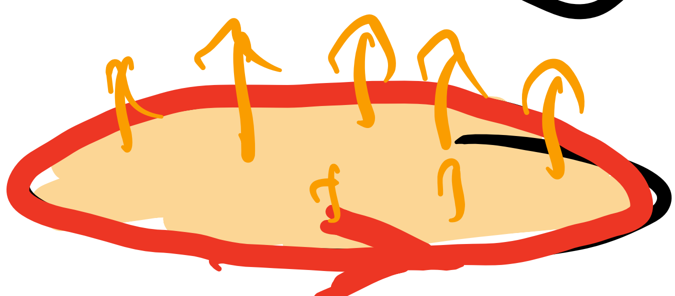
interior

A What happens if S is closed = has no boundary like a sphere

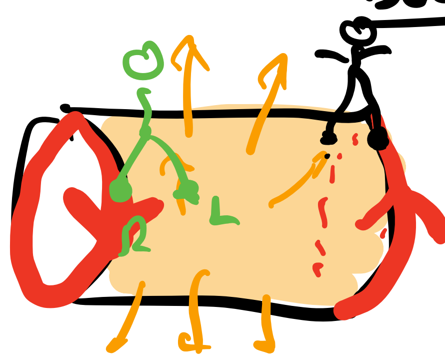


closed curve

no boundary = closed



has a boundary



$$\iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = 0$$

no boundary

$$\boxed{\text{The flux of the curl through a closed surface is zero}}$$

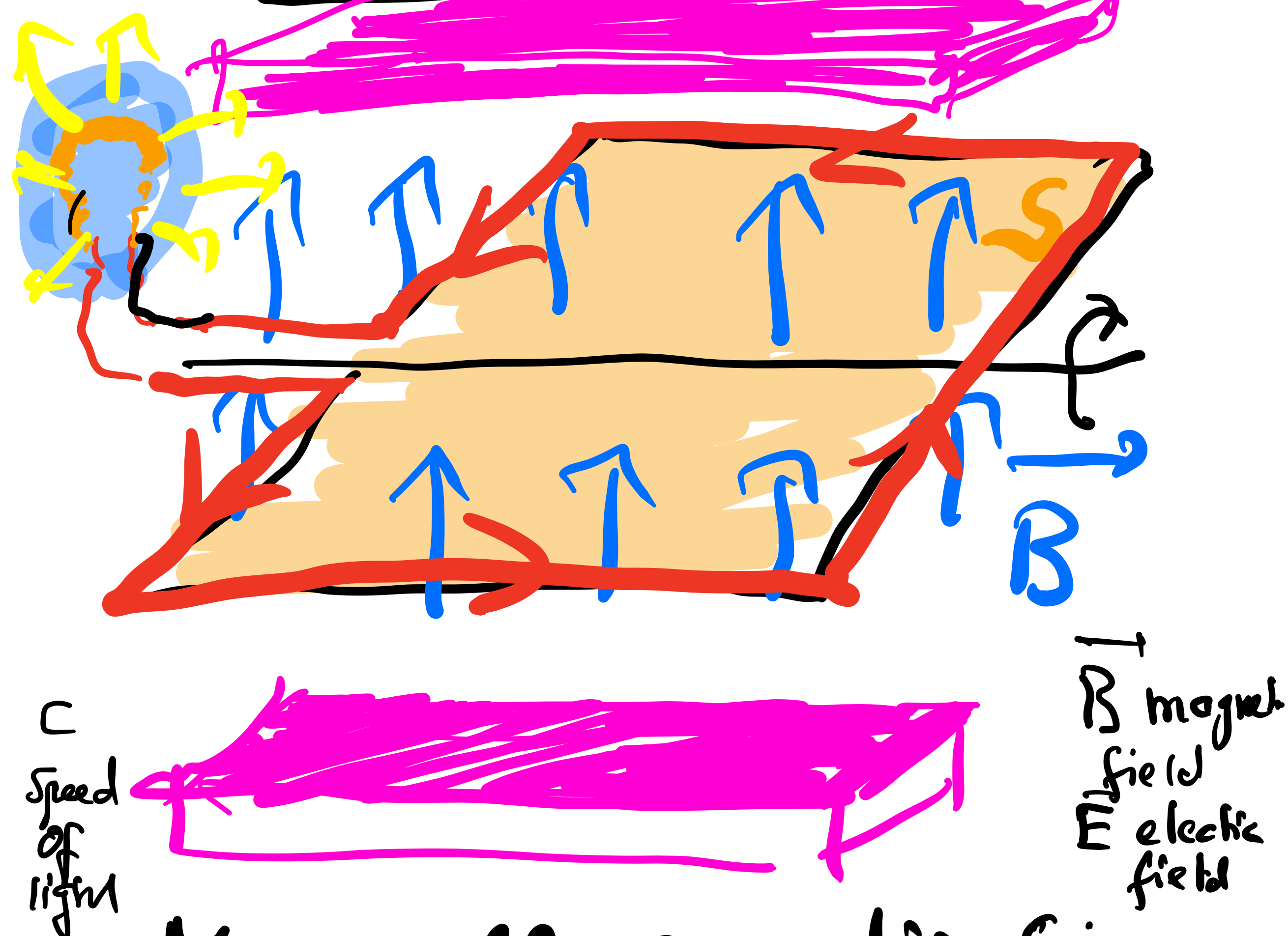
B) what happens if $\vec{F} = \nabla f$?

$$\iint_S \operatorname{curl}(\nabla f)\, d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

0 — Curl grad $f = 0$

0 — FTLI, closed curve

(4) Application



Maxwell equations:

$$\boxed{\begin{aligned} \mathrm{div}(\vec{E}) &= 4\pi\sigma \quad \text{charge} \\ \mathrm{div}(\vec{B}) &= 0 \\ \mathrm{curl}(\vec{E}) &= -\frac{1}{c}\frac{d}{dt}\vec{B} \\ \mathrm{curl}(\vec{B}) &= \frac{4\pi j}{c} + \frac{d}{dt}\vec{E} \end{aligned}}$$

$$\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$$ Change of magnetic flux

$$= \iint_S \frac{d}{dt} \vec{B} \cdot d\vec{S} \underset{\text{Maxwell}}{=} -c \iint_S \mathrm{curl}(\vec{E}) d\vec{S}$$

$$\underset{\text{Stokes}}{=} -c \int_C \vec{E} \cdot d\vec{r} = \text{electric energy.}$$

Maxwell was a student of Stokes and proving Stokes theorem was an exam problem.