Unit 22

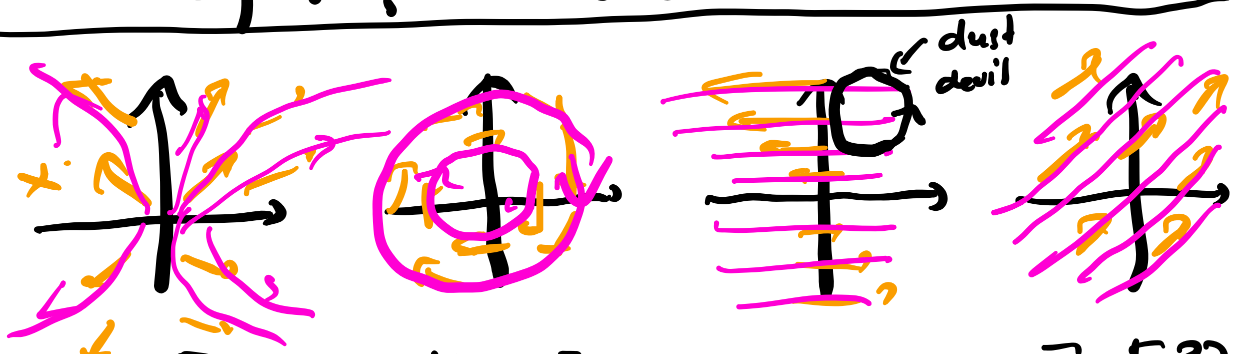
curl, div, flux

1) 2D curl

$\vec{F} = \begin{bmatrix} P \\ Q \end{bmatrix}$, $\text{curl}(\vec{F}) = Q_x - P_y$

$= \nabla \times \vec{F} = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \times \begin{bmatrix} P \\ Q \end{bmatrix}$

is a scalar.
It measures the vorticity of $\vec{F}$, "How curled is $\vec{F}$"



$\vec{F} = \begin{bmatrix} 2x \\ y \end{bmatrix}$ (A) $\quad$ $\vec{F} = \begin{bmatrix} y \\ -2x \end{bmatrix}$ (B) $\quad$ $\vec{F} = \begin{bmatrix} -y \\ 0 \end{bmatrix}$ (C) $\quad$ $\vec{F} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (D)

which are gradient fields? A D

$\text{curl}(\vec{F}) = 0$

A: $\begin{bmatrix} 2x \\ y \end{bmatrix} = \nabla f$, for $f = x^2 + \frac{y^2}{2}$

D: $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \nabla f$, for $f = 3x + 4y$.

B and C are <u>not gradient field</u>.

$\operatorname{curl}\left(\begin{bmatrix} y \\ 0 \end{bmatrix}\right) \equiv 1$, $\operatorname{curl}\begin{bmatrix} y \\ -2x \end{bmatrix} \equiv -3$

Green's theorem justifies the picture

in Europe $\operatorname{curl}(F) = \operatorname{rot}(\vec{F})$

2) 3D curl

Define $\operatorname{curl}\left(\begin{bmatrix} P \\ Q \\ R \end{bmatrix}\right) = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$

is again a vector field.

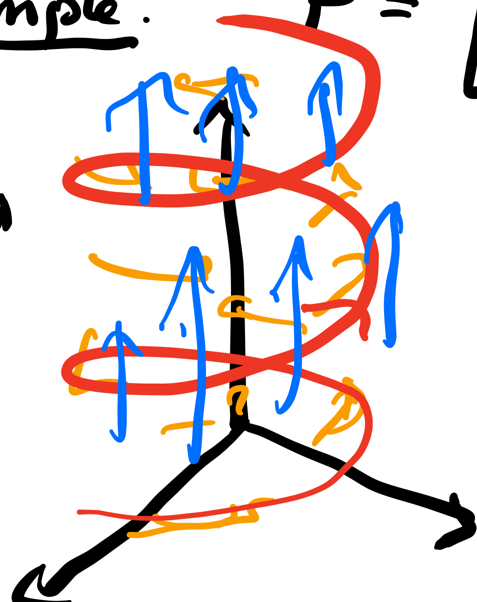
Better to see: $\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$

Nabla calculus: $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$

curl($\vec{F}$) tells you around which axis $\vec{F}$ rotates.
The $|\text{curl}(F)|$ is the amount of rotation.

<u>Example:</u> $\vec{F} = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$

"tornado"



Flow lines

curl($\vec{F}$)

$$\text{curl}(\vec{F}) = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Paddle wheel picture!

Special case HW

If $\vec{F} = \nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$

then

$$\operatorname{curl}(\vec{F}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using Clairaut.

If $\vec{F} = \nabla f$ then

$$\operatorname{curl}(\vec{F}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\vec{F}$ is called <u>irrotational</u>.

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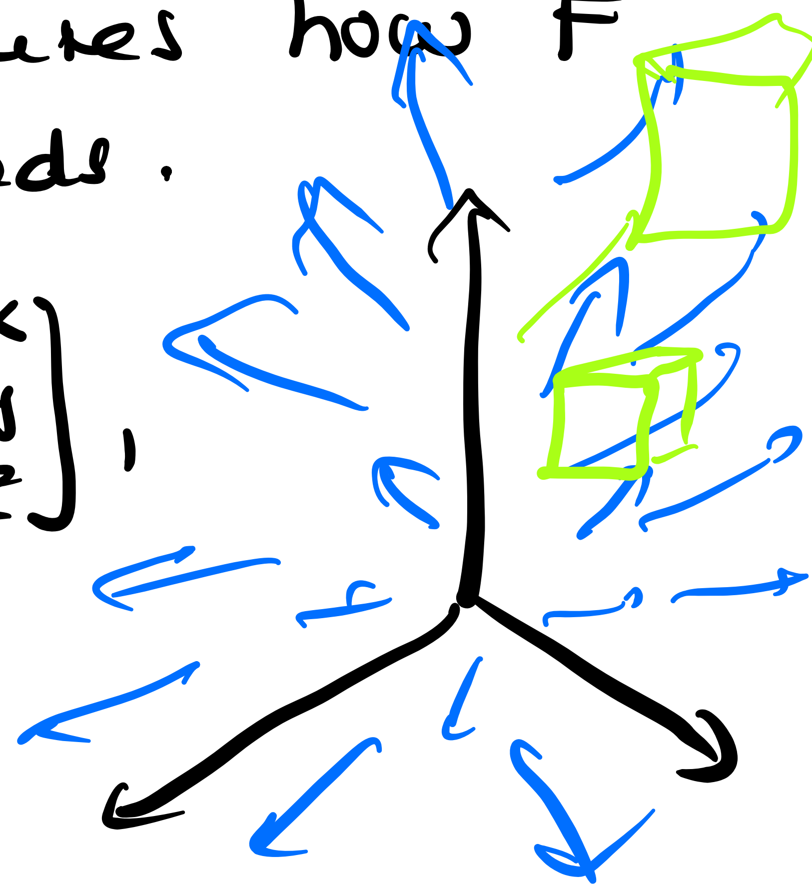
3) $\boxed{\text{Divergence}}$

$$\operatorname{div}\left(\begin{bmatrix} P \\ Q \\ R \end{bmatrix}\right) =$$

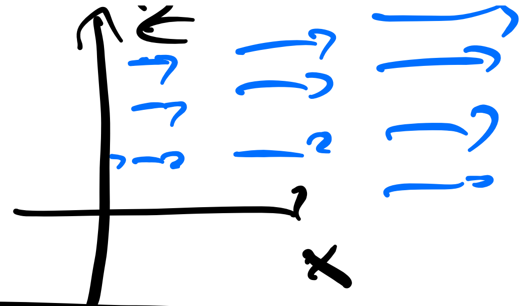
$$P_x + Q_y + R_z$$

measures how $\vec{F}$ expands.

$$\vec{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$



$$\vec{F} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$



If $\operatorname{div} \vec{F} = 0$, then $\vec{F}$ is called **incompressible**

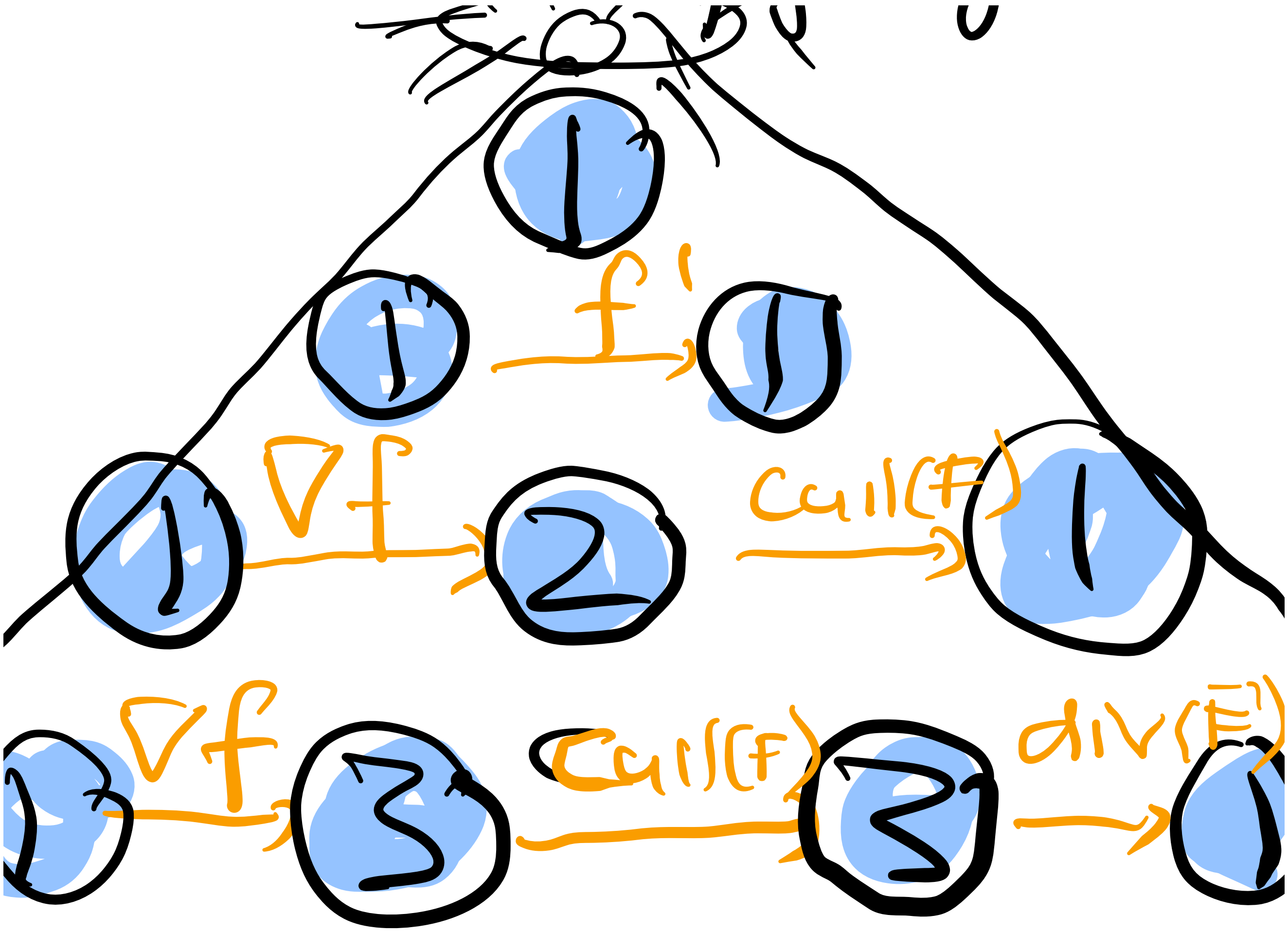
$\operatorname{div} \vec{F}$ tells how much "field" is generated at a point.

→ **Gauss**:

mass density

$$\boxed{\operatorname{div}(\vec{F}) = 4\pi\sigma}$$

Vector of gravity



4 Identities

HW

A $\boxed{\operatorname{curl}(\operatorname{grad} f) = \vec{0}}$

(B) $\boxed{\operatorname{div}[\operatorname{curl}(\vec{F})] = 0}$

$$\operatorname{div}\begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

Candy crush (Clairaut)

$\operatorname{div}\left(\begin{bmatrix} P \\ Q \end{bmatrix}\right) = P_x + Q_y$

is defined too

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$

$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$

$\operatorname{grad} f = \nabla f$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\omega}) = 0$

$\operatorname{div} \operatorname{curl}(\vec{F})$

$= \nabla \cdot (\nabla \times \vec{F})$

$= 0$

curl grad $f = \nabla \times \nabla f$

$= 0$ $\qquad$ $\vec{V} \times \vec{V} = 0$

It is not true for example that

$\nabla \times \vec{F}$ is

perpendicular to $\vec{F}$

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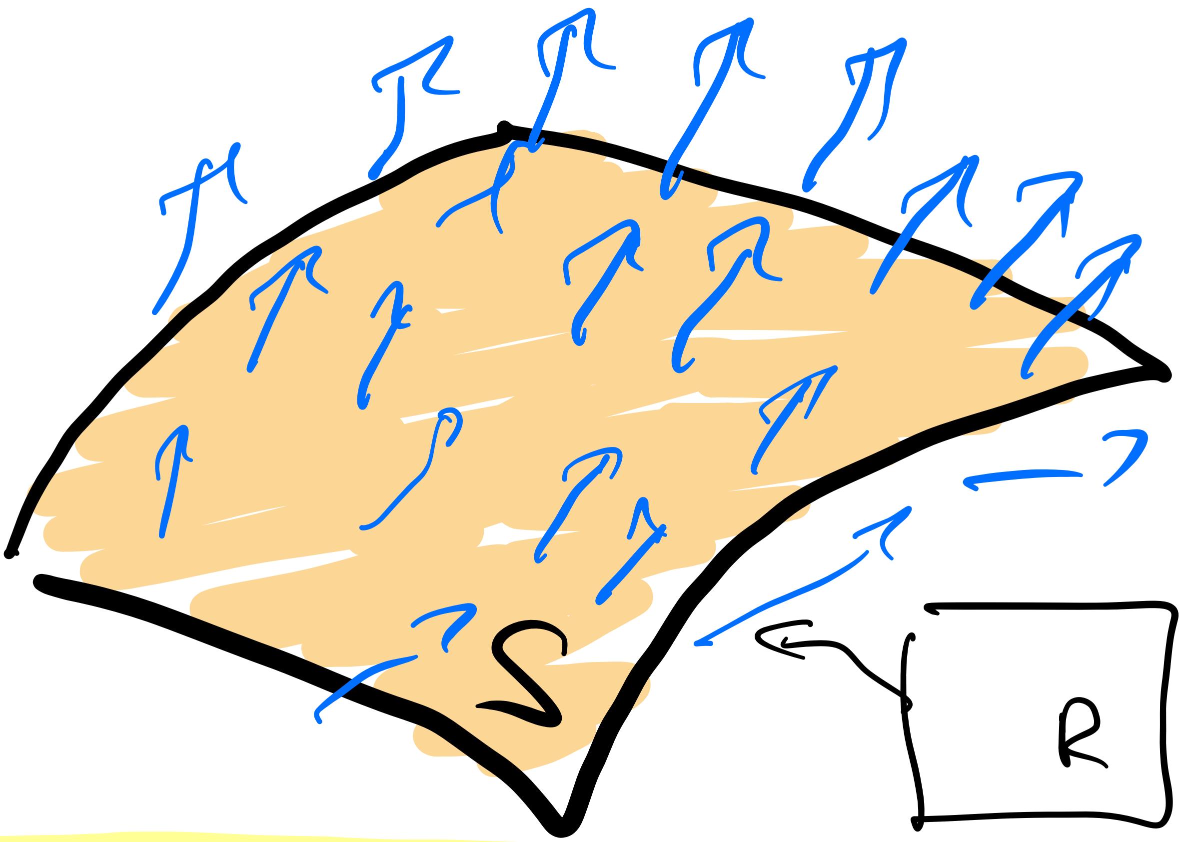
⑤ Flux integral



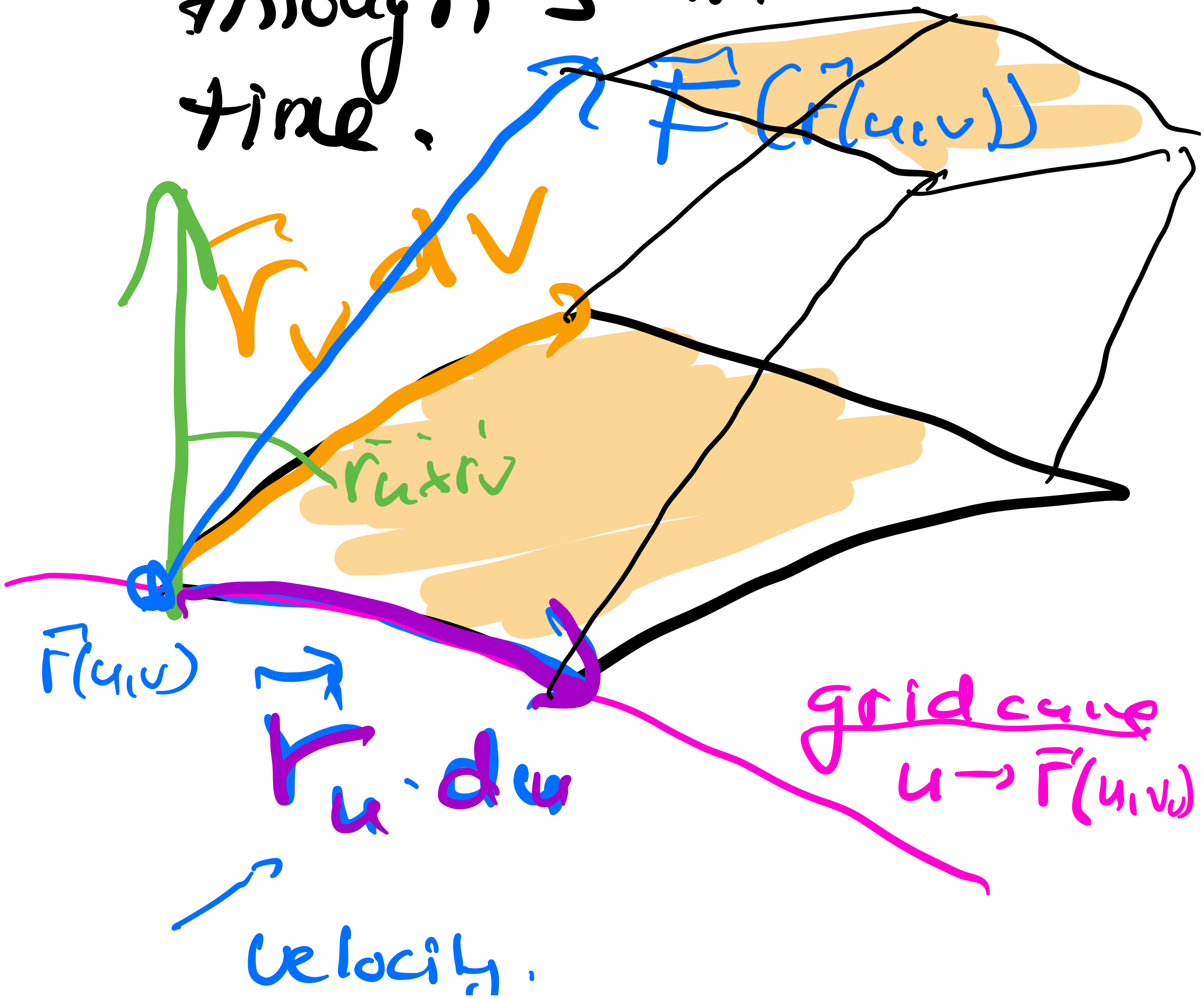
$$\iint_S \vec{F} \cdot d\vec{S}$$ is defined as

volume

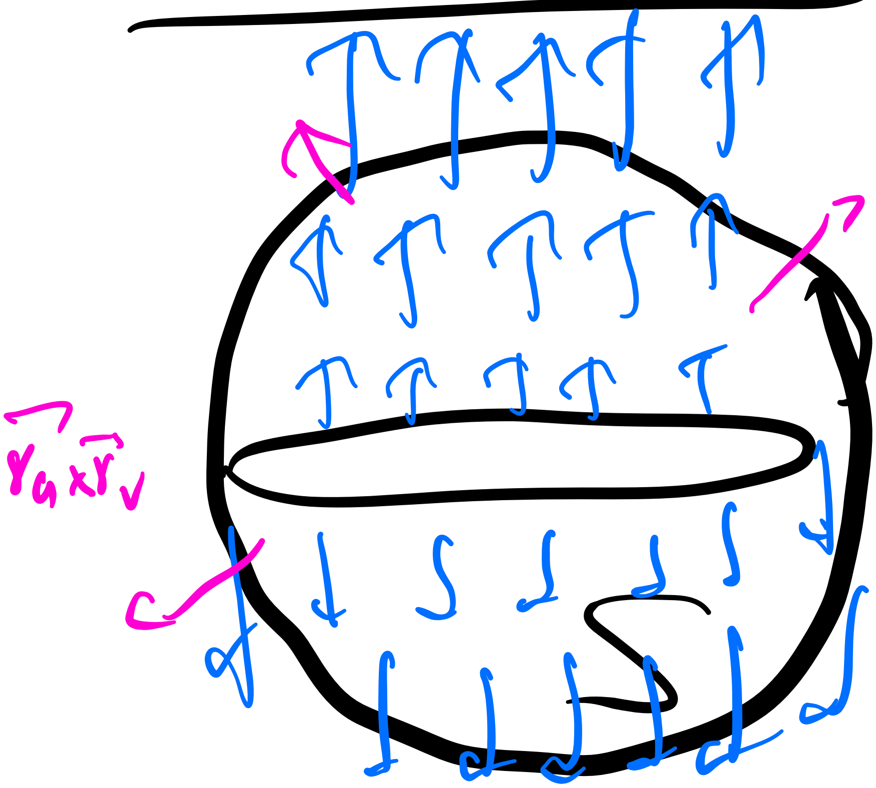
$$\iint \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

R

This **flux integral** measures how much field $\vec{F}$ passes through S in unit time.



Example:



$\vec{F}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$

$x^2 + y^2 + z^2 = 1$

unit sphere oriented outwards

$\vec{r}_u \times \vec{r}_v$ points outward, What is the flux of $\vec{F}$ through S?

$$\vec{r}(\varphi,\theta) = \begin{bmatrix} \cos\theta \sin\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\int_0^{2\pi}\int_0^{\pi} \begin{bmatrix} 0 \\ 0 \\ \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \cdots \\ \cdots \\ \sin\varphi\cos\varphi \end{bmatrix} d\varphi\, d\theta$$

$$r_\varphi \times r_\theta = \sin\varphi \begin{bmatrix} \cos\theta \sin\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$= \int_0^{2\pi}\int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi\, d\theta = \frac{2}{3} \cdot 2\pi = \boxed{\frac{4\pi}{3}}$$

$$-\frac{\cos^3(\varphi)}{3}\Big|_0^{\pi} = \frac{2}{3}$$

This is the volume.

This is not an accident!

— Thursday.