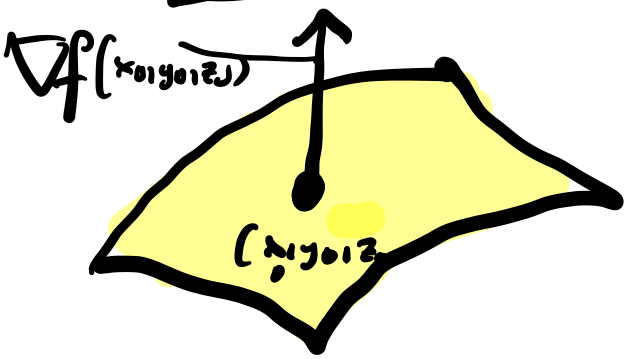
Unit 12 Tangent spaces

1 Gradient theorem



$f(x,y,z)=c$
level surface
(x_0,y_0,z_0)

Theorem: $\nabla f(x_0,y_0,z_0)$ is $\perp$ to the level surface

E: $f(x,y,z) = ax+by+cz = d$

$$\nabla f(x,y,z) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \vec{n}$$

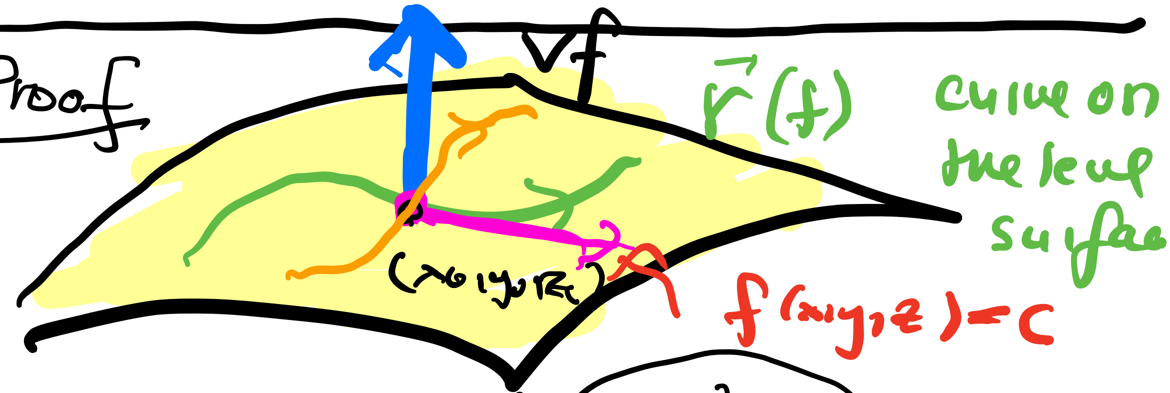
The theorem tells that $\vec{n}$ is perpendicular to the plane.

E: $f(x,y,z) = x^2 - y^2 + z = 1$

hyperbolic paraboloid.

$$\nabla f = \begin{bmatrix} 2x \\ -2y \\ 1 \end{bmatrix}, \quad \nabla f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Proof



What is $\frac{d}{dt} f(\vec r(t)) = 0$

$f(\vec r(t)) = c$ because we are on the surface.

Use the chain rule:

$$\boxed{\nabla f(\vec r(t))} \cdot \boxed{\vec r\,'(t)} = 0$$

↑ tangent to the surface

Because this is true for all curves through (x_0, y_0, z_0), $\vec n$ is $\perp$ to the surface.

QED

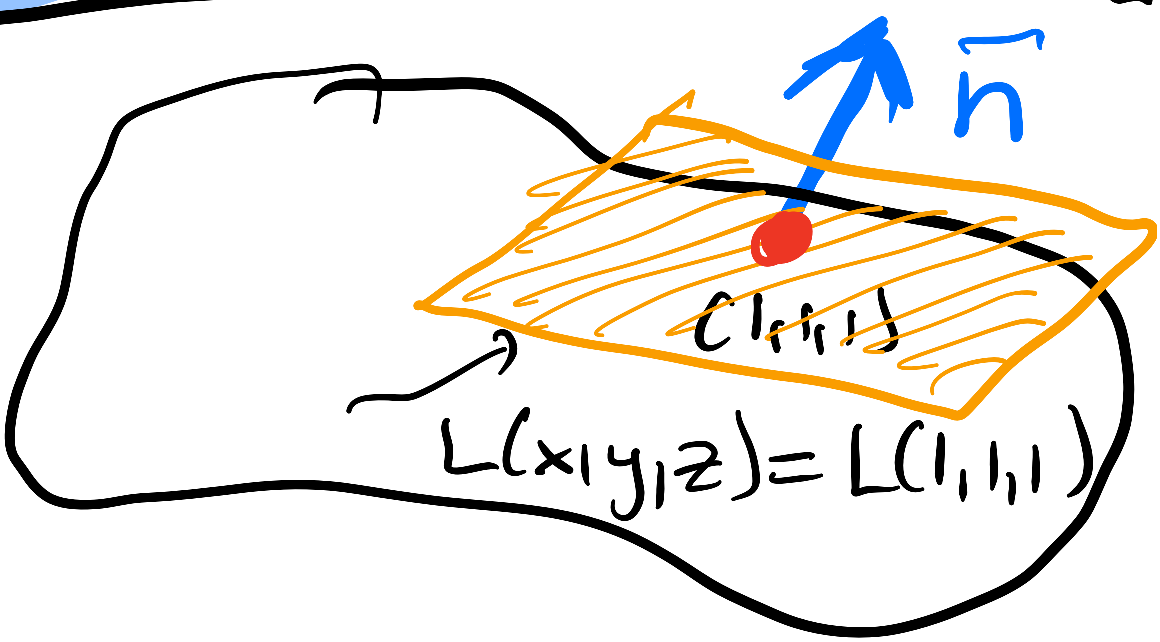
② Tangent planes

Problem

Find the tangent plane to

$$x^2z^2+y^2z^2+2x^2y^2=4$$

at the point $(1,1,1)$



$$\nabla f(1,1,1) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$ax + by + cz = d$$

$$\nabla f(1,1,1) = \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix}$$

$$\boxed{6x + 6y + 4z = 16}$$

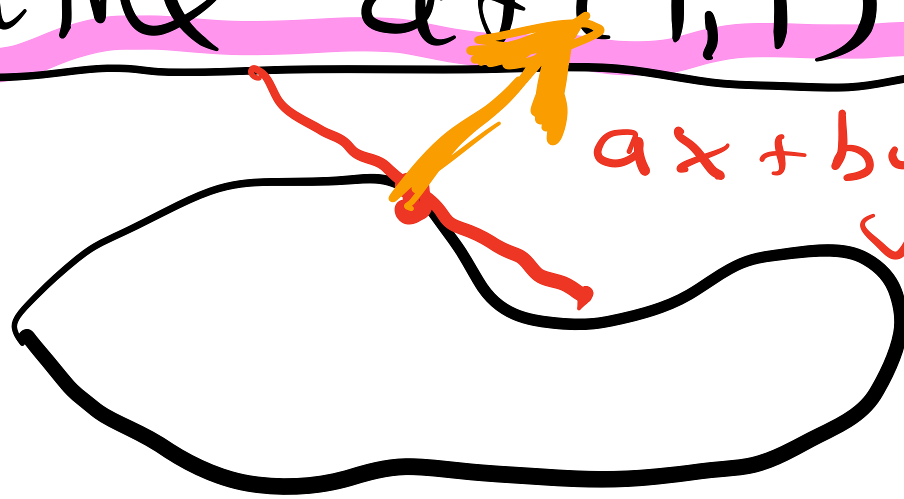
Plug in (1,1,1)

We have made use of the fact that the tangent plane and the surface have the same normal direction

3 Tangent lines

$$x^2y + y^3x^5 = 2$$

Find the tangent line at $(1,1)$



$ax + by = d$

The same idea:

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$\boxed{7x + 4y = 15}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 2$$

at $(x_0, y_0) = (2, 3)$

$$\cancel{y = \sqrt{2 - \frac{y^2}{9}}}$$

don't solve for y even if you can, this time.

$$\nabla f = \begin{bmatrix} 1 \\ \frac{6}{9} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\boxed{x + \frac{6}{9} y = 2 + 2 = 4}$$

4 Directional derivative

f function of 2 or 3 variables

$\vec{v}$ direction = unit vector

$|\vec{v}| = 1$

$$\boxed{D_{\vec{v}} f = \nabla f \cdot \vec{v}}$$

what happens if

$$\hat{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D_{\hat{v}} f = f_x$$

partial derivatives are special case

∇f is a vector

$D_{\hat{v}} f$ is a scalar

$D_{\vec{v}} f$ is the rate of change of f when you move in the $\vec{v}$ direction.

It allows you to estimate how f is like if you go into the direction $\vec{v}$

5. **Steepest ascent**

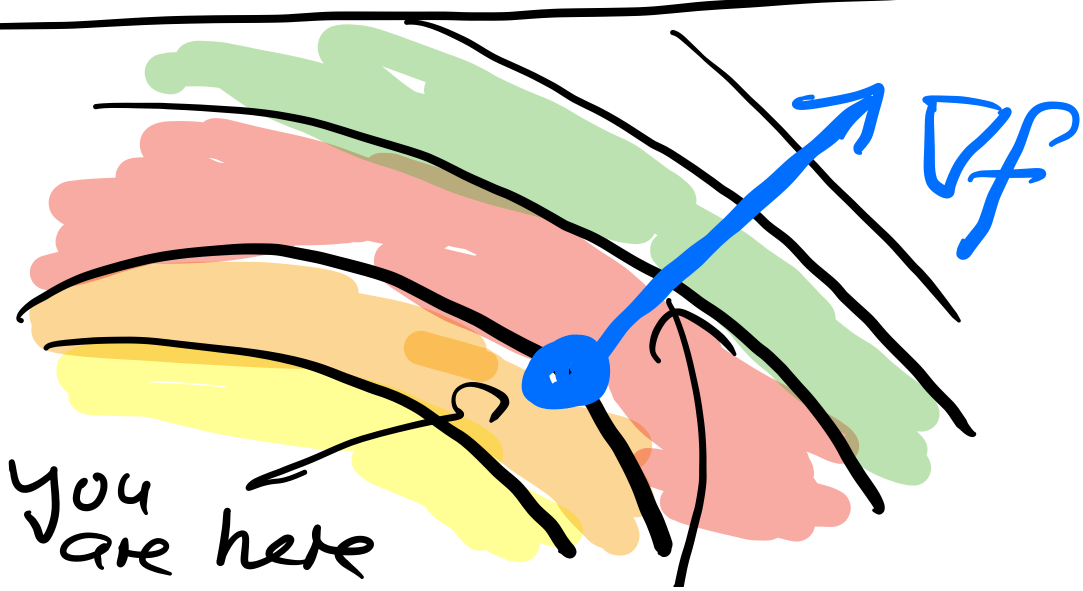
The direction of the gradient is the direction of steepest ascent

$$|D_{\vec{v}} f| \le |\nabla f|\,|\vec{v}|$$
$$= |\nabla f|$$

But if $\vec{v} = \dfrac{\nabla f}{|\nabla f|}$

then

$$D_{\vec{v}} f = \frac{\nabla f \cdot \nabla f}{|\nabla f|}$$
$$= \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$



you go in
this direction
to reach to a
top value

Machine
learning principle.