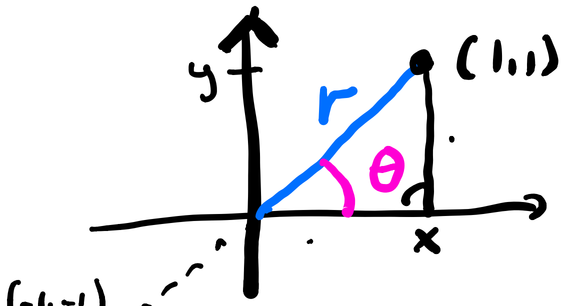
Unit 6

θ Theta
ϑ Theta

1) Polar coordinates

$r \geq 0$
$0 \leq \theta < 2\pi$

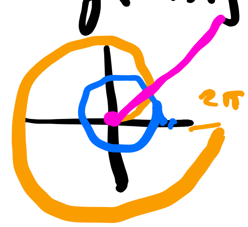


$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$

$r = \sqrt{x^2+y^2}$

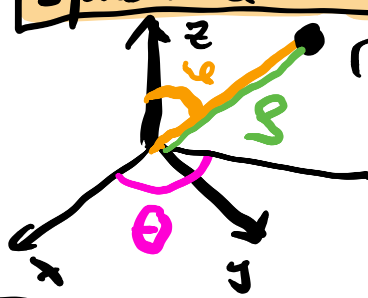
only up to π $\longrightarrow$ $\theta =$ "$\arctan\left(\frac{y}{x}\right)$"
$(= \arg(x+iy))$

$\boxed{r = 2}$ circle
$\boxed{r = \theta}$ spiral
$\boxed{\theta = \frac{\pi}{4}}$ half line

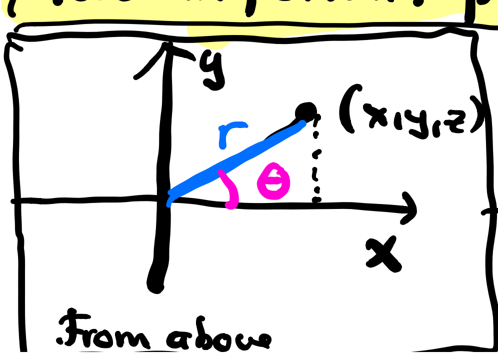


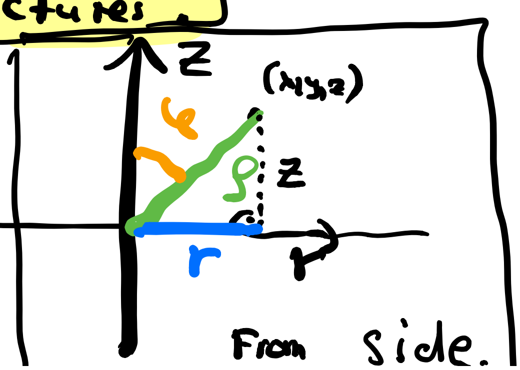
2) spherical coordinates

φ = Phi
ϕ = Phi
ρ = Rho



Two important pictures!





$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}} \qquad \boxed{\begin{aligned} r &= \rho\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}}$$

$$\boxed{\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}}$$

φ and θ are called Euler angles.

$$\rho \geq 0, \qquad 0 \leq \theta < 2\pi$$
$$0 \leq \varphi \leq \pi$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

What is $\boxed{\rho = 9}$ sphere of radius 9

$\boxed{\rho\sin\varphi = 2}$ cylinder

$r =$ constant means: the distance to the z axis is constant

$$r = \sqrt{x^2+y^2} \qquad r = c \Rightarrow x^2+y^2 = c^2$$

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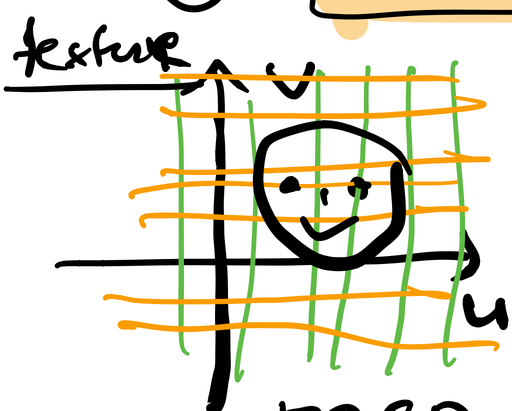
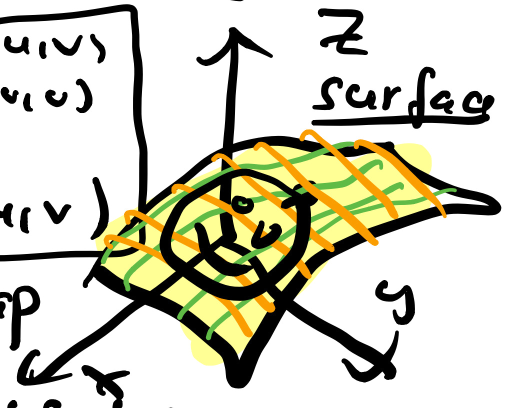
(3) Parametrization



$$\boxed{\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}}$$

uv-map



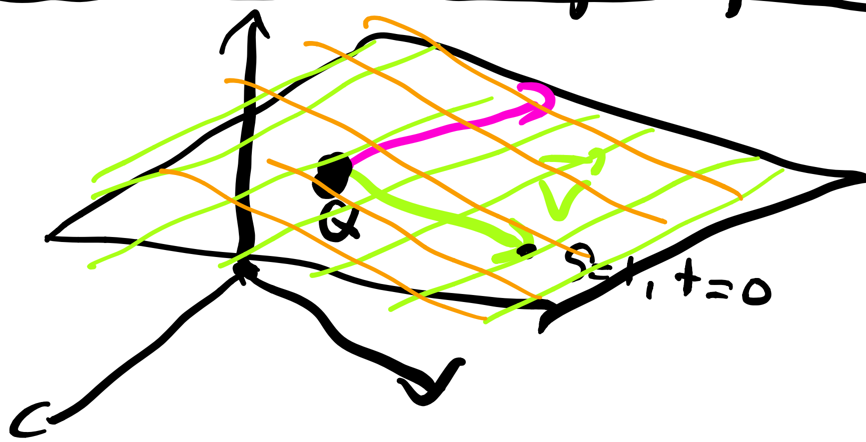
map

Parametrization

④ Examples (Basic)

a) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

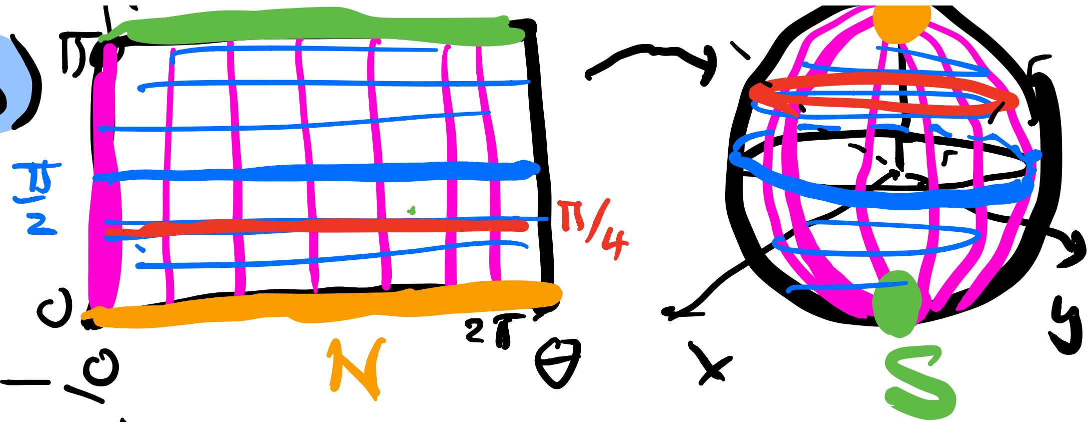
Parametrization of a plane



A plane is 2 dimensional. It only needs 2 parameters to determine a point.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s + 3t \\ 3 + s + 4t \\ 7 + s + t \end{bmatrix} = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

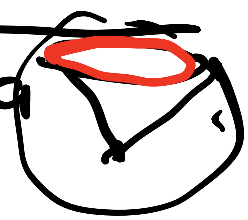
b)



Parametriz. of a sphere:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\varphi \cos\theta \\ \sin\varphi \sin\theta \\ \cos\varphi \end{bmatrix}$$

$\boxed{\varphi = \frac{\pi}{4}}$ this is a cone



$\boxed{\varphi = \frac{\pi}{4},\ \rho = 1}$ this is a circle on the sphere $\rho = 1$

On a sphere we need only 2 coordinates to determine a point: logitude and latitude.

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c) $z = f(x,y)$
graph.

How do we parametrize this?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

(i) $$z = \frac{x^2}{4} - \frac{y^2}{9}$$

which is a hyperbolic paraboloid

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ \frac{x^2}{4} - \frac{y^2}{9} \end{bmatrix}$$

(ii) $y - \sin xz + x = 0$

can be parametrized:

$$y = g(x,z) = \sin(xz) + x$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ \sin xz - x \\ z \end{bmatrix}$$

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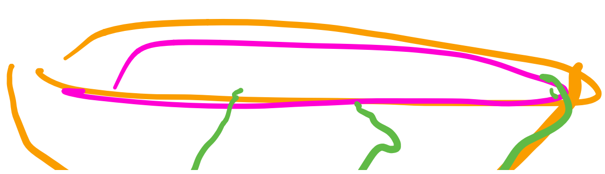
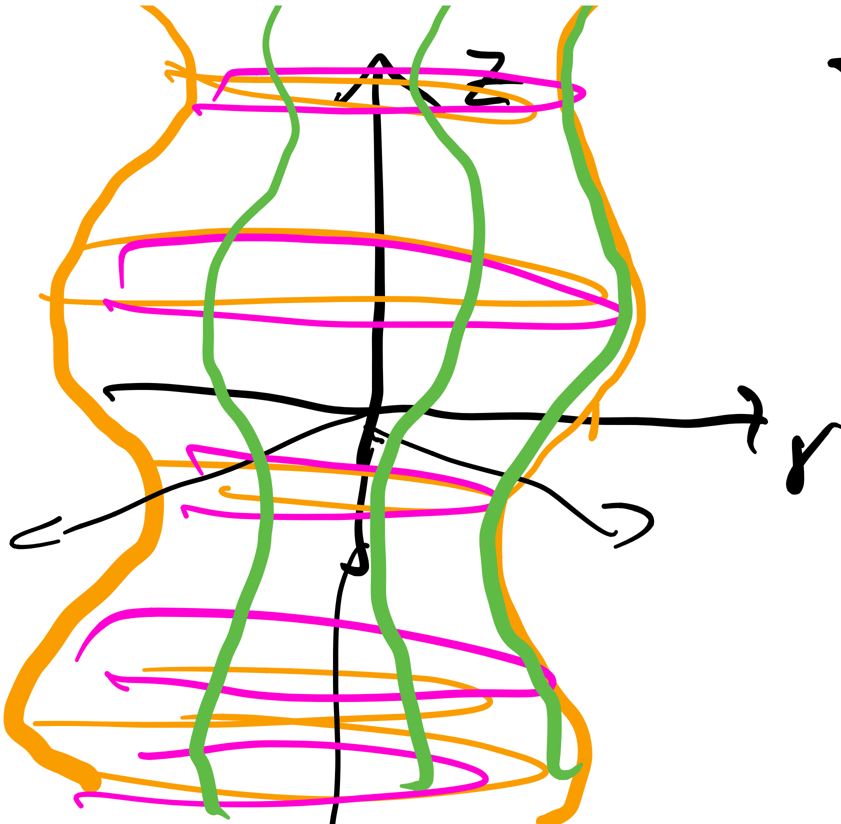
a) <u>Surface of revolution</u>

$$r = g(z)$$

$$r = (2 + \sin z)$$

we spin around the graph $r = 2 + \sin z$ around the z axis





z and θ are good coordinat

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} g(z)\cos\theta \\ g(z)\sin\theta \\ z \end{bmatrix}$$

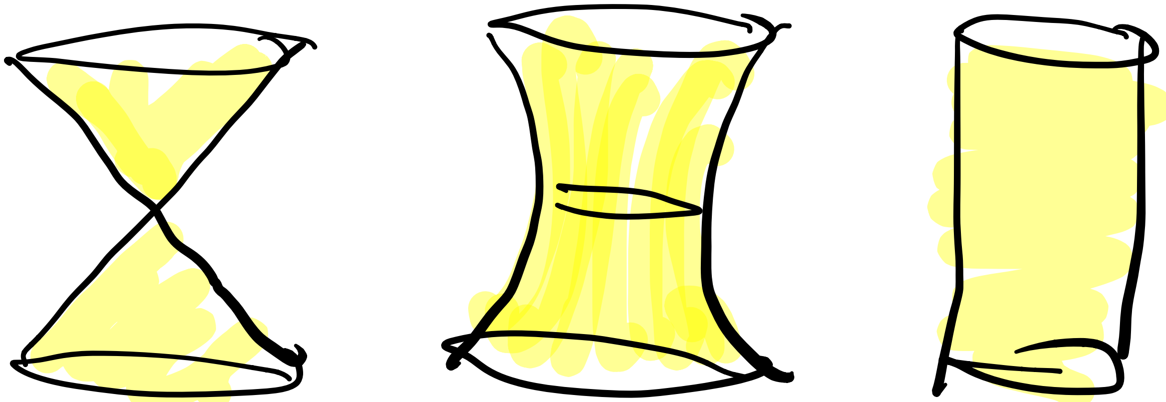
Keep in mind these 4 examples!

Practice:

a) Parametrize $\boxed{x^2+y^2=z^2}$

b) Parametrize $\boxed{x^2+y^2-z^2=1}$

c) Parametrize $\boxed{x^2+y^2=1}$



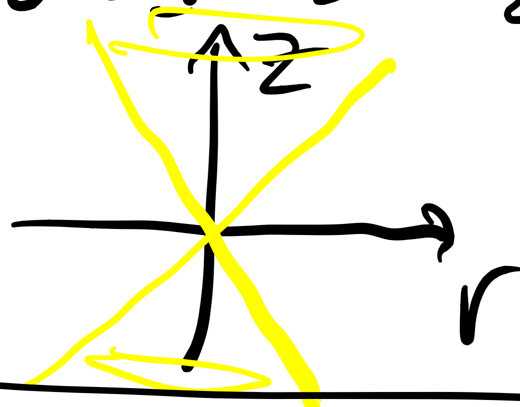
a) b) c)

a) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ \sqrt{x^2+y^2} \end{bmatrix}$$

only the upper part.

Cone

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z\cos\theta \\ z\sin\theta \\ z \end{bmatrix}$$



$r = z$

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b) Hyperboloid

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{z^2+1}\,\cos\theta \\ \sqrt{z^2+1}\,\sin\theta \\ z \end{bmatrix}$$

$$x^2+y^2-z^2=1$$

$$r^2 = z^2+1$$

$$r = \sqrt{z^2+1}$$

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c) Cylinder

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1\cos\theta \\ 1\sin\theta \\ z \end{bmatrix}$$

(5) $\boxed{\text{More examples}}$

→ Mathematica demo