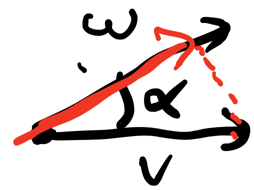
Unit 3

3.1 Dotproduct

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\alpha$$

$$\boxed{P_{\vec{w}}\, \vec{v}} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \frac{\vec{w}}{|\vec{w}|}$$



Reminders.

3.2 Cross product

$\vec{v} \times \vec{w}$ is a vector

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3\cdot 7 - 5\cdot 4 \\ 4\cdot 1 - 7\cdot 2 \\ 2\cdot 5 - 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \\ 7 \end{bmatrix}$$

$$\begin{matrix} 2 & & 1 \\ 3 & & 5 \end{matrix}$$

$$[2, 3, 4] \times [1, 5, 7] = [1, -10, 7]$$

$$\det \begin{bmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

"determinant"

Def

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

Cross product.

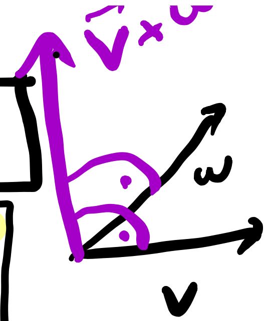
$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{matrix} 1 & & 3 \\ 1 & & 2 \end{matrix}$$

3.3 $\boxed{\text{Properties}}$

(i) $\boxed{\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}}$

(ii) $\vec{v}$ is perpend to $\vec{v} \times \vec{w}$
$\vec{w}$ is perpend to $\vec{v} \times \vec{w}$



$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} =$$

$$v_1 v_2 w_3 - v_1 v_3 w_2 + v_2 v_3 w_1 - v_2 v_1 w_3 + v_3 v_1 w_2 - v_3 v_2 w_1 = 0$$

Candy crush

(iii) $\boxed{|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \cdot \sin\alpha}$

we first check

verified by AI.

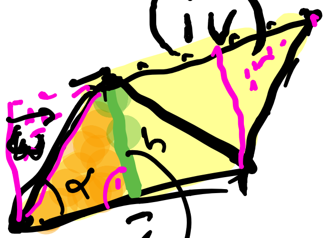
Cauchy Binet 1812

$$|\vec{v} \times \vec{w}|^2 + (\vec{v} \cdot \vec{w})^2 = |\vec{v}|^2 |\vec{w}|^2$$

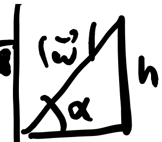
why does this argue the formula?

$$|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - |\vec{v}|^2 |w|^2 \cos^2(\alpha)$$
$$= |\vec{v}|^2 |\vec{w}|^2 (1 - \cos^2(\alpha))$$
$$= |\vec{v}|^2 |w|^2 \sin^2\alpha$$

Take square root : QED.

(iv) 

$|\vec{v} \times \vec{w}|$ is the are[a] of a <u>parallelogram</u> spanned by $\vec{v} \times \vec{w}$



$\sin\alpha = \frac{h}{|\vec{w}|}$

$h = \boxed{|\vec{w}| \cdot \sin\alpha}$ height of ▱.

Area: $h \cdot$ base

$\underbrace{|\vec{v}|}_{\text{base}} \underbrace{|\vec{w}| \sin\alpha}_{h}$

$$\frac{|\vec{v} \wedge \vec{w}|}{2} = \boxed{\begin{array}{l} c \cdot b \cdot \sin\alpha \\ = a \cdot b \cdot \sin\gamma \end{array}} = a \cdot c \cdot \sin\beta$$

divide through

$$\frac{c \sin\alpha}{a \sin\gamma} = 1 \qquad \boxed{\frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}}$$

3.4 Use of cross product

- <u>Area of triangle</u> without <u>trigonometry</u>
- <u>Distances</u> between objects like lines.

- in higher dim n

$\vec{v} \wedge \vec{w}$ $\qquad \wedge$ wedge

is an object of dim $\binom{n}{2}$

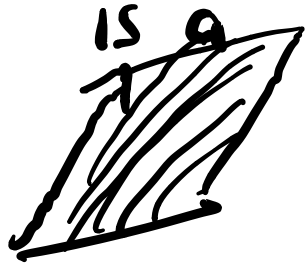
$= \dfrac{n(n-1)}{2}$.

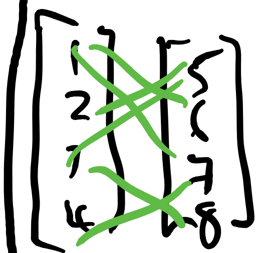
$n=3 \qquad \binom{3}{2} = \dfrac{3(2)}{2} = 3$

$n=2 \qquad \binom{2}{2} = 1$

$$\begin{bmatrix}3\\2\end{bmatrix} \times \begin{bmatrix}1\\4\end{bmatrix} = 10$$

In 2 dim, the cross product is a scalar.





in dim $n=4$

$$\binom{4}{2} = \frac{4\cdot 3}{2} = 6$$

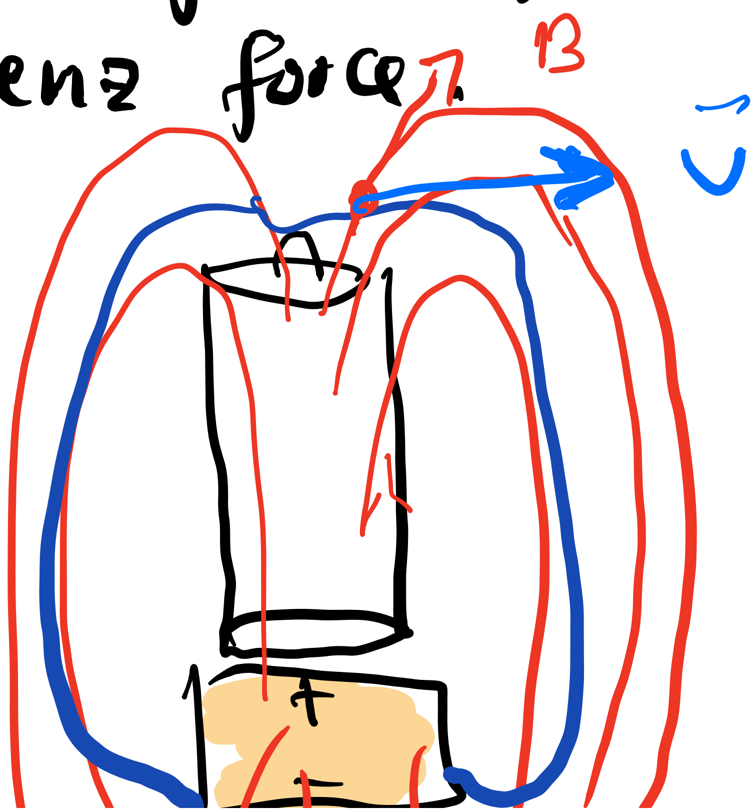
$\vec{v} \wedge \vec{w}$ has 6 dim.

- **Magnetism:**

$$\vec{v} \times \vec{B} = \vec{F}$$

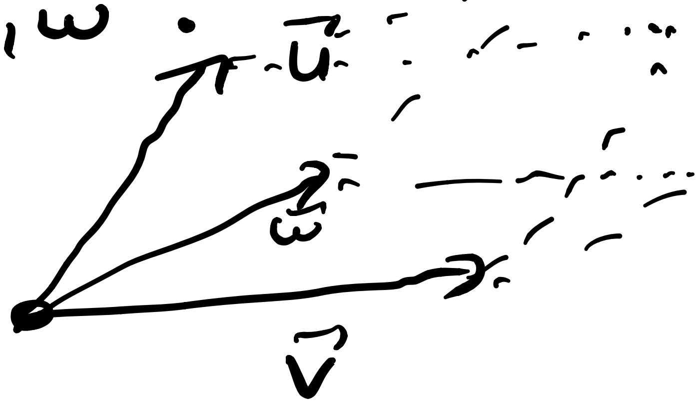
$\vec{B}$ magnetic field

Lorenz force



3.5 $\boxed{\text{Triple scalar product}}$

$\vec{u}\cdot(\vec{v}\times\vec{w})$ is called the <u>triple scalar product</u> of $\vec{u}, \vec{v}, \vec{w}$.



(i) $|\vec{u}\cdot(\vec{v}\times\vec{w})|$ is the <u>volume</u> of the <u>parallelepiped</u> spanned by $\vec{u}, \vec{v}, \vec{w}$.

(ii) If $\vec{u} \cdot (\vec{v} \times \vec{w}) > 0$

then $\vec{u}, \vec{v}, \vec{w}$

define a right

handed system

$\vec{i}, \vec{j}, \vec{k}$ form a right

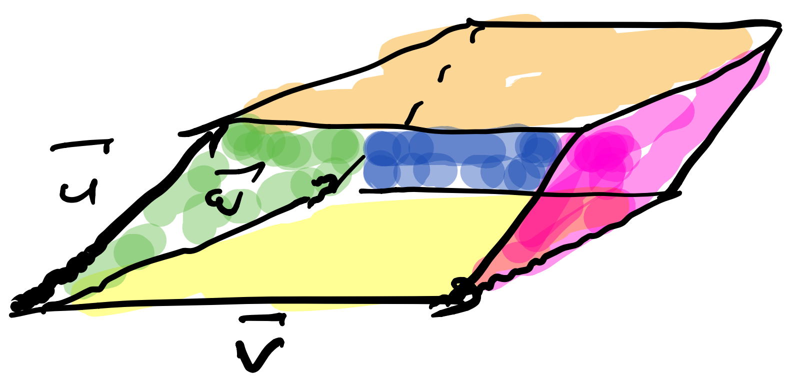
handed sys

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = 1$$

Why do we get the volume?

Parallelepiped.



(if all three vectors are in a plane then the triple scal. product is 0)

volume = base area $\cdot h$

$= |\vec{v} \times \vec{w}| \cdot h$

$= |\vec{v} \times \vec{w}| \cdot |\vec{u}| |\cos \alpha|$

$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

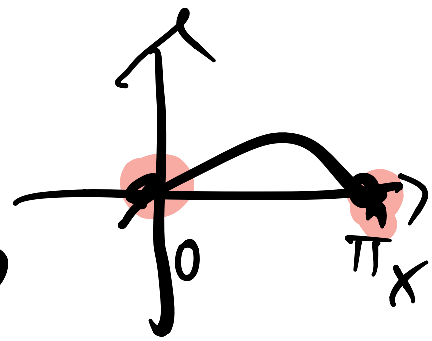
if $\vec{u}$ and $\vec{v} \times \vec{w}$ are on different sides, then

this is negative
Then you have a left handed system.

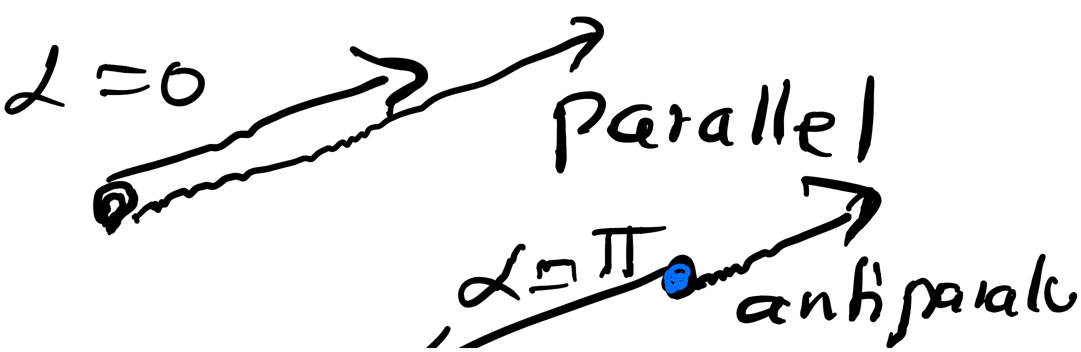
$$\vec{V} \times \vec{w} = 0$$

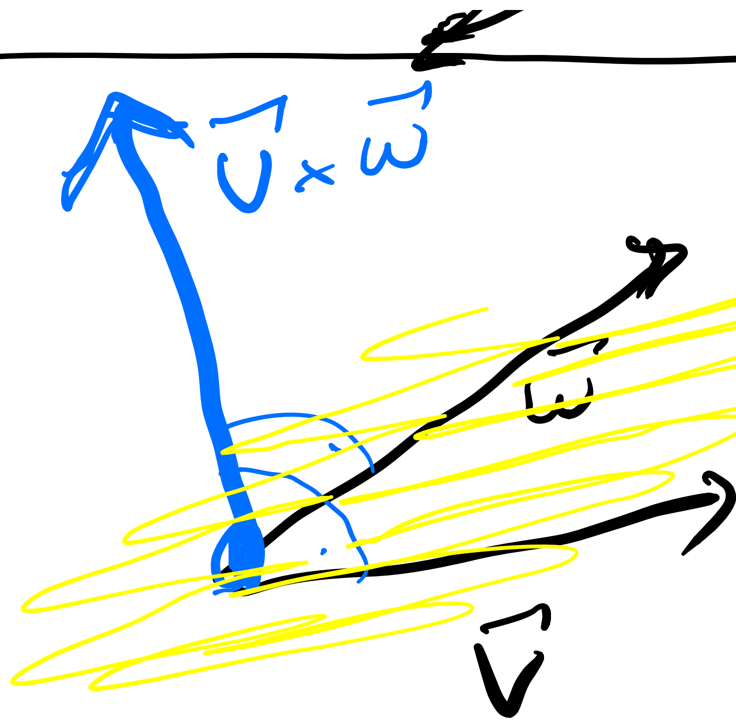
if $|\vec{v}| = 0$

or $|\vec{w}| = 0$

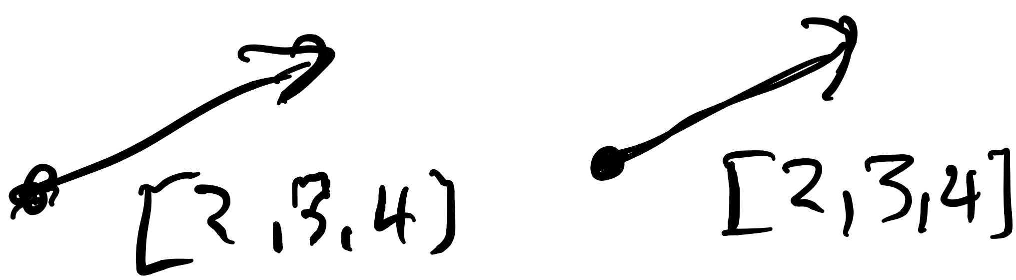


or $\sin\alpha = 0$

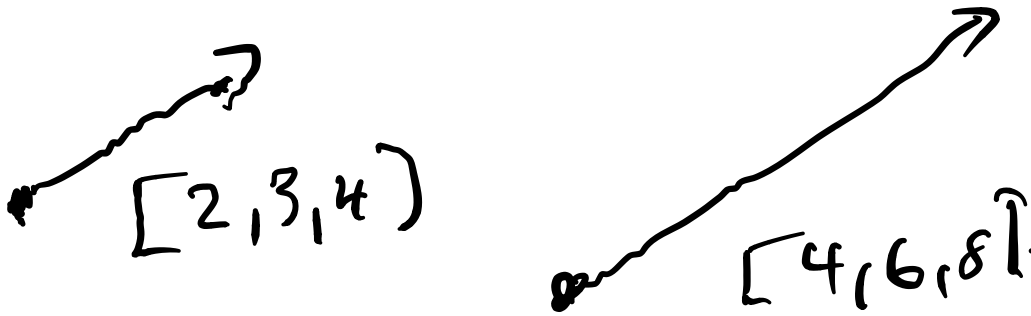




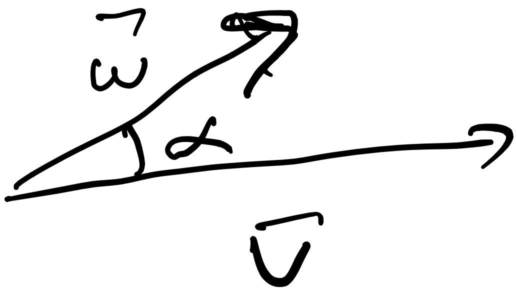
$\vec{v} \times \vec{w}$ is normal to the plane spanned by $\vec{v}$ and $\vec{w}$



"Same vectors"



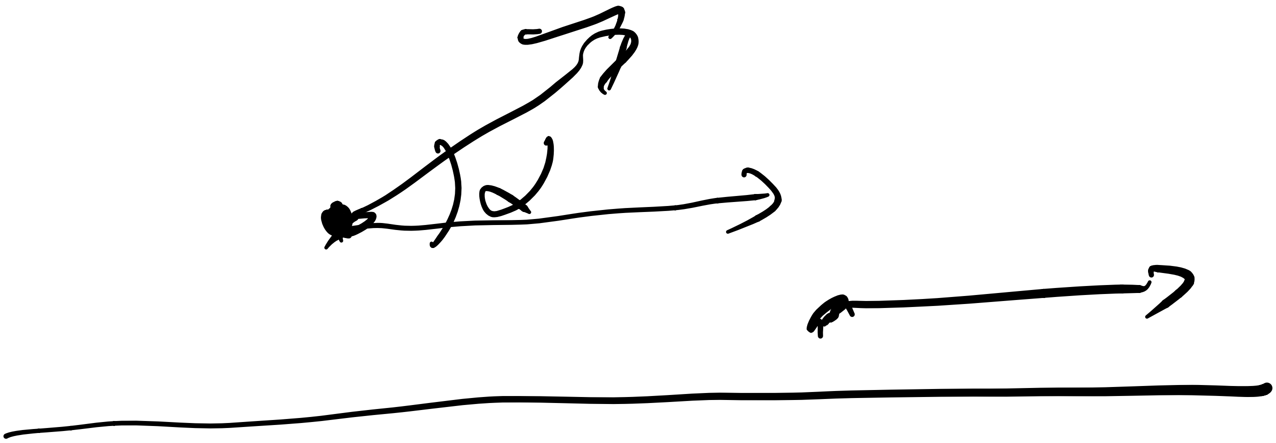
parallel



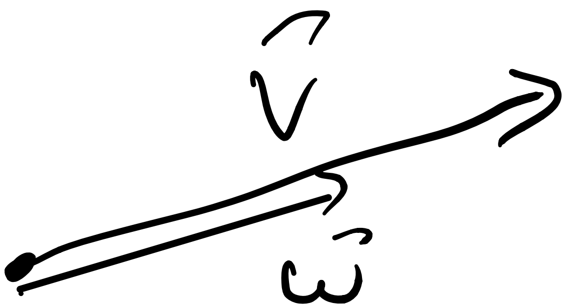
$\alpha \to 0$ then

$\vec{v}$ $\vec{w}$ become

parallel.



How do we get vectors perpend to $\vec{v}$ and $\vec{w}$ if they are parallel



We can use the crossed produc

$$[3, 4, 17]$$

$$\downarrow$$

$$[-4, 3, 0]$$

$$[0, -17, 4]$$