unit 2 Vectors and dot product

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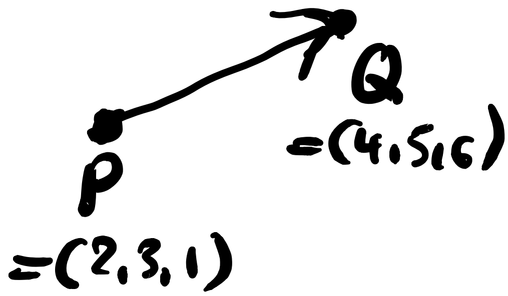

P, Q define a vector

$\vec{PQ} = \vec{v}$

$$\vec{PQ} = [4-2, 5-3, 6-1]$$
$$= [2, 2, 5] = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

components

magnitude: $|\vec{PQ}| = d(P,Q)$

= length $= \sqrt{2^2+2^2+5^2}$

$\vec{v} \neq \vec{0}$ $= \sqrt{33}$

direction: $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{[2,2,5]}{\sqrt{33}}$

unit vector.

Does Every vector have a direction and magnitude?

No! $[0,0,0] = \vec{0}$

does not have a
direction!

$$[2,3,4]+[1,1,1]=[3,4,5]$$

$$[2,3,4]-[1,1,1]=[1,2,3]$$

$$3\,[2,3,4]=[6,9,12]$$

Operations which
work like with numbers!

$$\cancel{\frac{\vec{v}}{\vec{w}}}$$

no division
no multiplical

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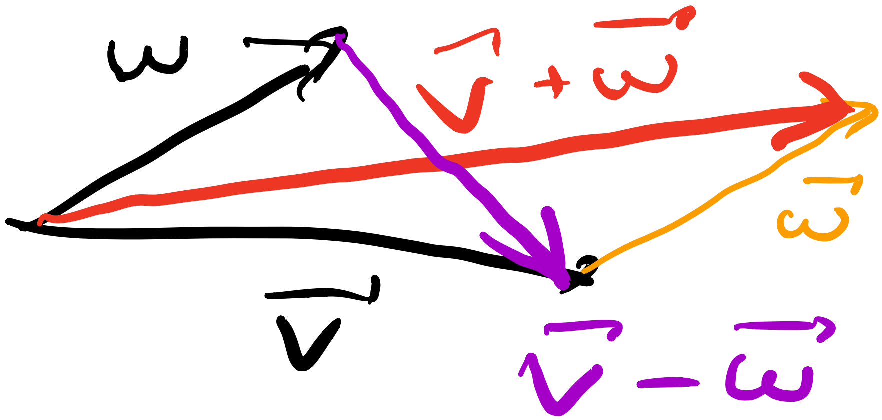
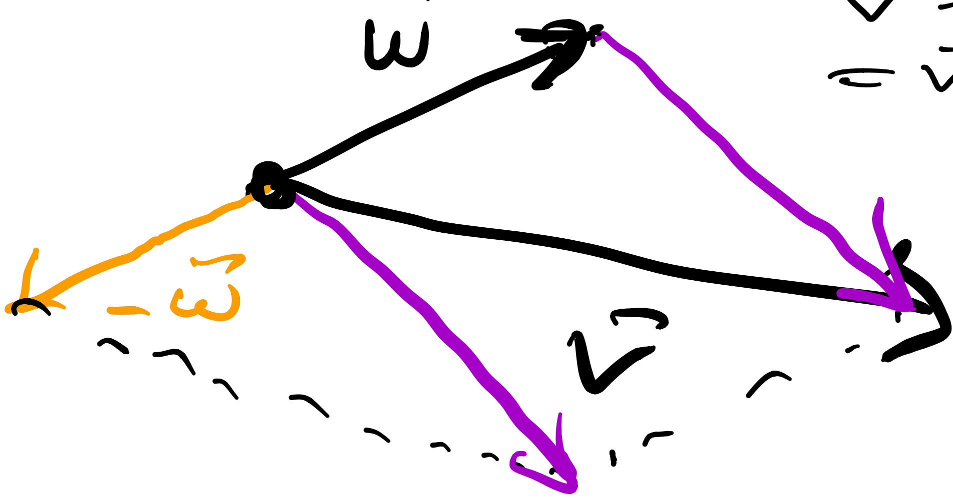
one calls this a
vector space



geometric interpretation



$\vec{v}-\vec{w}$
$= \vec{v} + (-\vec{w})$

② $\boxed{\text{Dot product}}$

$$[2,3,4] \cdot [4,5,6]$$

$$2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 = 47$$

dot product or

Scalar product or

inner product

$$[3, 1, 0] \cdot [1, 1, 1]$$

$$= 4$$

$$\vec{v} \cdot \vec{v} = [v_1, v_2, v_3] \cdot [v_1, v_2, v_3]$$

$$= v_1^2 + v_2^2 + v_3^2$$

$$= |v|^2$$

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$$

③ Kauchy Schwarz

$$|\vec{v}\cdot\vec{w}| \leq |\vec{v}|\,|\vec{w}|$$

Proof: First assume

$|\vec{w}| = 1$

otherwise divide by

$|\vec{w}| = 0$ or $|\vec{w}| = 0$

then the claim is clear.

$a = \vec{v}\cdot\vec{w}$ FOIL

≥ 0

$|\vec{v} - a\vec{w}|^2 = (\vec{v} - a\vec{w})\cdot(\vec{v} - a\vec{w})$

$\vec{v}\cdot\vec{v} - 2a(\vec{v}\cdot\vec{w}) + a^2(\vec{w}\cdot\vec{w})$

$$|v|^2 - 2a^2 + a^2$$

$$|v|^2 - a^2 \geq 0$$

$$|v|^2 \geq a^2$$

$$|v| \geq |\vec{v} \cdot \vec{w}|$$

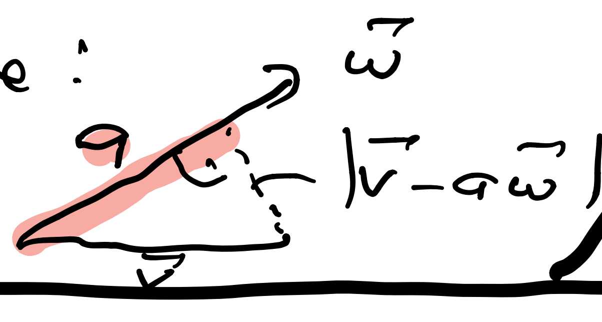
$$|w|\,|v| \geq |v \cdot \vec{w}|$$

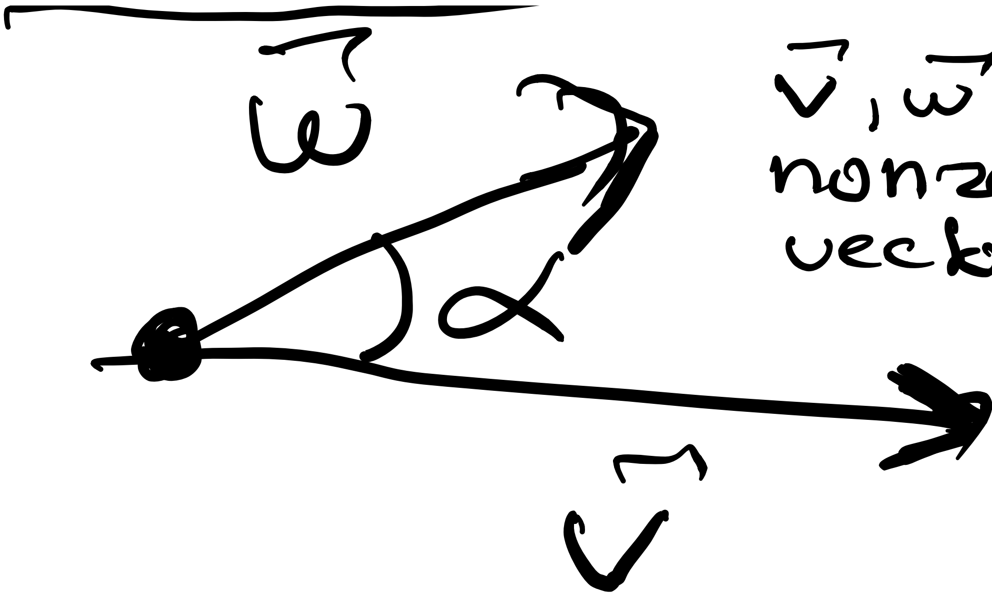
QED

quod erat demonstrand

quite easily done

This proof is motivated by a picture:





$\vec{v}, \vec{w}$ nonzero vectors

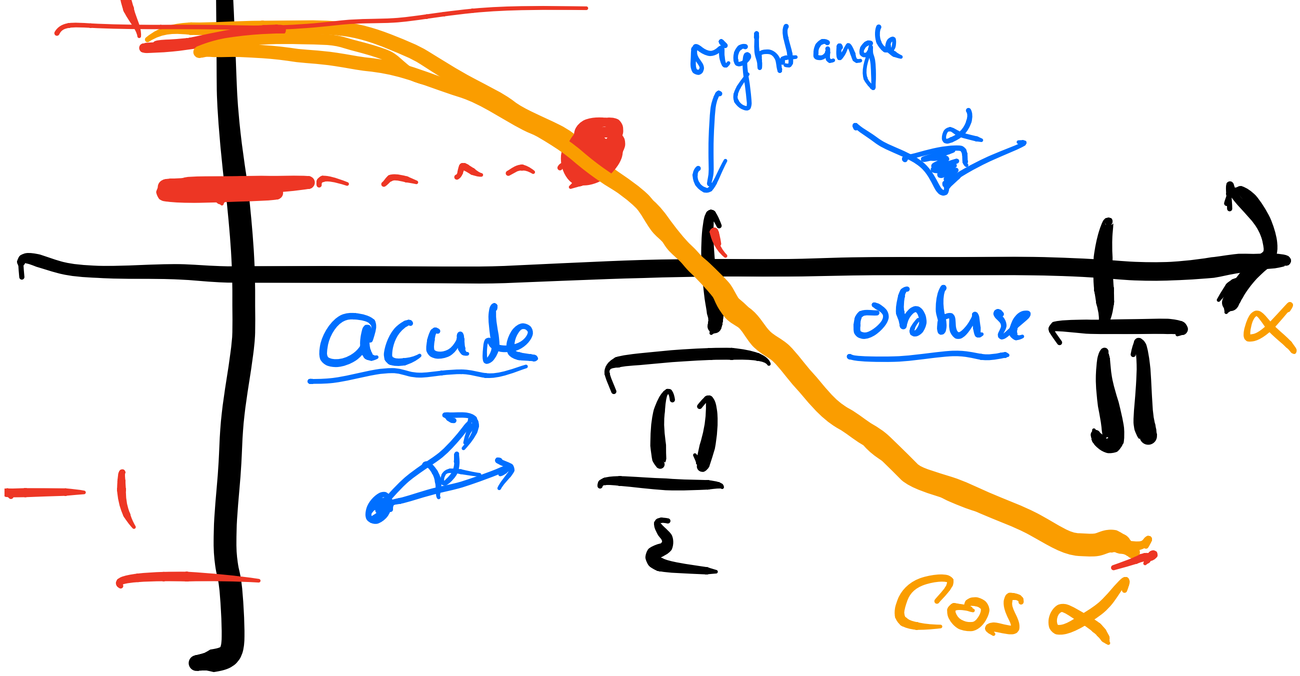
$$\boxed{\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}}$$

why can we do this?

Answer:

By Cauchy-Schwarz!

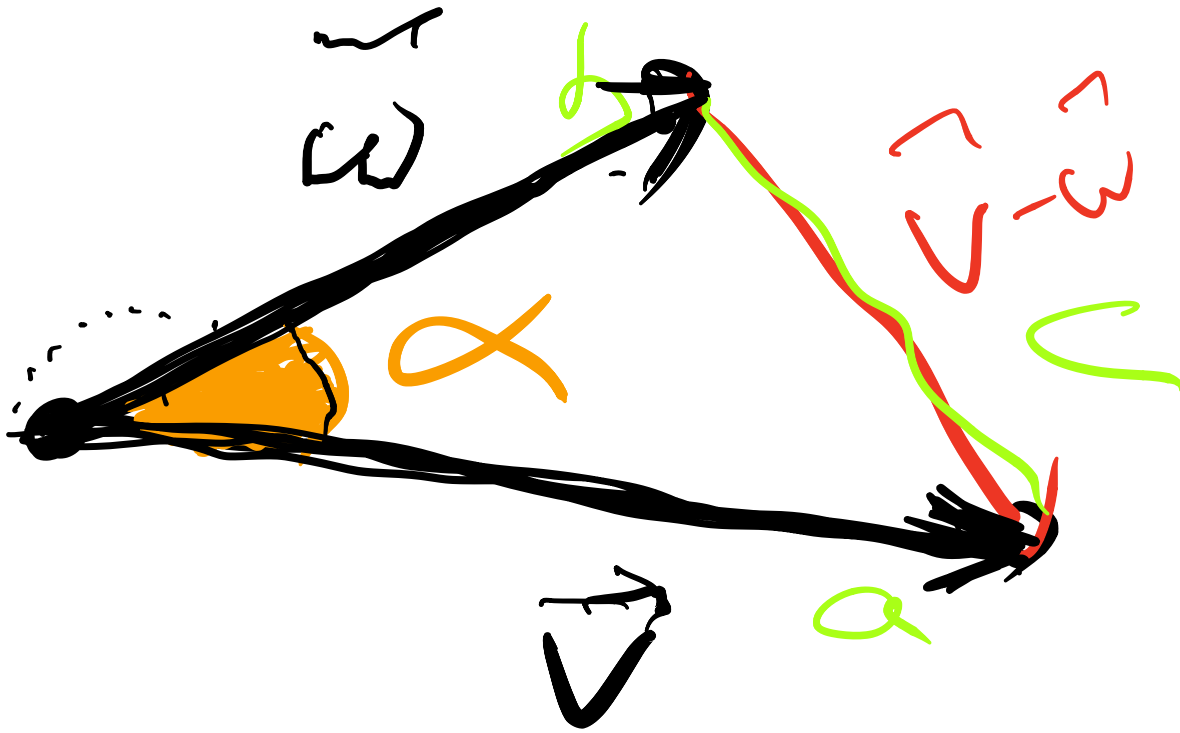
$$\boxed{-1 \leq \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|} \leq 1}$$



- $\cos\alpha > 0$ positive corr.
- $\cos\alpha < 0$ negative corr
- $\cos\alpha = 0$ uncorrelated

This value $\cos\alpha$ is also known in statistics, if $\vec{v}, \vec{w}$ are data. It is the <u>correlation coefficient</u>!

4 Pythagoras



$$(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$

$$\underset{\text{FOIL}}{=} |\vec{v}|^2 - 2\vec{v}\cdot\vec{w}$$

$$+ |w|^2$$

rewrite

$$c^2 = a^2 - 2\vec{v}\cdot\vec{w} + b^2$$

Now use $\vec{v}\cdot\vec{w}$

$$= |v||w|\cos\alpha$$
$$= a\,b\cos\alpha$$

$$\boxed{c^2 = a^2 + b^2 - 2\,ab\cos\alpha}$$

Al Khashi

cos-formula.

Especially!

Take $\alpha = \frac{\pi}{2}$

$= 90^{\circ}$

$\cos\alpha = 0$

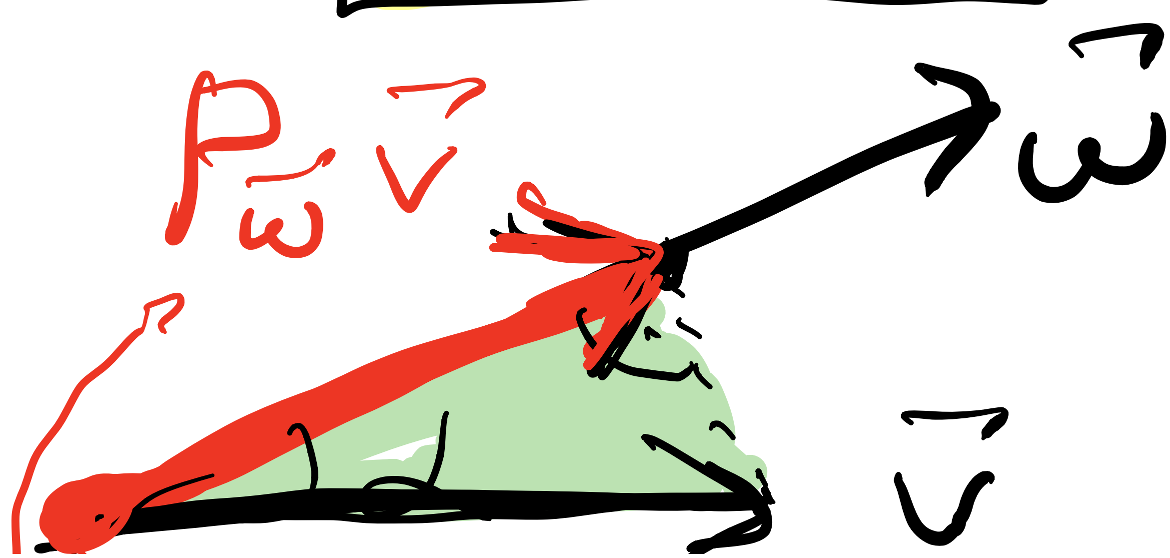
$$\boxed{c^2 = a^2 + b^2}$$

Pythagoras theorem

$$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

$$\boxed{\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\alpha}$$

5 **Projection**



vector projection

has length

$|\vec{v}| \cdot \cos\alpha$

length of red vector

$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$

def. sn.

SOH
CAH
TOA

multiply with a direction $\frac{\vec{w}}{|\vec{w}|}$

$$\boxed{\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}\,\frac{\vec{w}}{|w|}=\vec{p}}$$