7/22/2021 SECOND HOURLY PRACTICE 5 Maths 21a, O.Knill, Summer 2021

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The function $f(x,y) = x^6 + y^6$ has exactly one minimum.

2) T F The gradient $\nabla f(1,1,1)$ is tangent to the surface $x^2 + y^2 - z^2 = 1$.

3) T F The directional derivative $D_{\vec{v}} f(0,0)$ is positive if the unit vector $\vec{v}$ and the vector $\nabla f(0,0)$ form an acute angle.

4) T F If $f_x(0,0) = f_y(0,0) = 0$ and $f_{xx}(0,0) > 0$, $f_{yy}(0,0) > 0$, $f_{xy}(0,0) < 0$, then $(0,0)$ is a local minimum of f.

5) T F If $f_x(0,0) = f_y(0,0) = 0$ and $f_{xx} > 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) < 0$, then $(0,0)$ is a saddle point.

6) T F The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \nabla f(\vec{r}\,'(t))$ for any function $f(x,y,z)$ and any curve $\vec{r}(t) = [x(t), y(t), z(t)]$.

7) T F The point $(0,1)$ is a maximum of $f(x,y) = x + y$ under the constraint $g(x,y) = x = 1$.

8) T F The function $u(x,y) = \sqrt{x^2 + y^2}$ solves the partial differential equation $u_x^2 + u_y^2 = 1$.

9) T F Let $f(x,y) = x^3 y^2$. At $(0,0)$ and every direction $\vec{v}$ we have $D_{\vec{v}} f(0,0) = 0$.

10) T F The identity $f_{xyx} = f_{yxy}$ holds for all smooth functions $f(x,y)$.

11) T F The integral $\int_0^x \int_0^y s^2 t^2 \, ds dt$ is positive if $x > 0$ and $y > 0$.

12) T F If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \times \vec{r}(u,v) = 0$.

13) T F We have $|D_{[1,0]} f(0,0)| \leq |\nabla f(0,0)|$.

14) T F Any smooth function $f(x,y)$ has a critical point inside the region $0 \leq x^2 + y^2 < 1$.

15) T F The surface area of a surface $\vec{r}(u,v) = [2u, 3v, 0]$ with (u,v) in a disk $R = \{u^2 + v^2 \leq 1\}$ is given by the integral $\int\int_R 6 \, dudv$.

16) T F The Lagrange multiplier λ of a critical point of $f(x,y)$ is positive if the function $f(x,y)$ is positive there.

17) T F The equation $f_{xy} = f^2 f_x + f_y$ is an example of a partial differential equation.

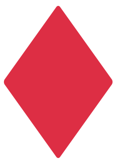
18) T F If $(0,0)$ is a critical point of $f(x,y)$ and $f_{xy}(0,0) = 0$ and $f_{xx}(0,0)$, $f_{yy}(0,0)$ have different signs, then $(0,0)$ is a saddle point.

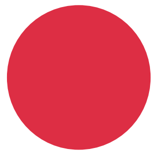
19) T F If $f(x,y)$ has the critical point $(0,0)$, then $(f(x,y))^4$ has the critical point $(0,0)$ too.

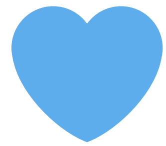
20) T F If $f(x,y)$ is the linear function $f(x,y) = 3x + y - 3$, then $g(x,y) = |\nabla f(x,y)|$ is constant.

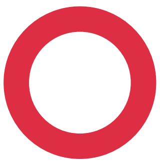
Problem 2) (10 points) No justifications are needed

a) (6 points) Some hate them, some love them. Any way, last Monday was **Emoji day**. Match the following Emoji with the integrals. There is an exact match.

A 

B 

C 

D 

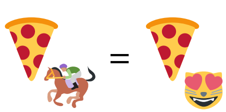
E 

F 

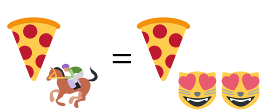
Enter A-F	Area integral
	$\int_0^{2\pi} \int_0^{1+\cos^6(4\theta)} \; r dr d\theta$
	$\int_0^{2\pi} \int_0^{(\theta-\pi/2))^2} \; r dr d\theta$
	$\int_{-1}^1 \int_{\lvert x\rvert-1}^{1-\lvert x\rvert} 1 \; dxdy$
	$\int_0^{2\pi} \int_0^1 r \; dr d\theta$
	$\int_0^{2\pi} \int_0^{1+\cos^4(5\theta/2-1)} \; r dr d\theta$
	$\int_0^{2\pi} \int_3^4 r \; dr d\theta$

b) (4 points) Match the following **partial differential equations**. The pizza Emoji is a function and the rest of the Emoji's are variables. We transport hamburgers in the heat wave:

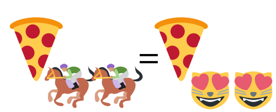
Fill in a)-d)	Name
	Transport
	Burgers
	Heat
	Wave



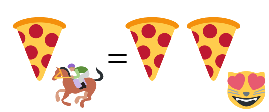
a)



b)



c)

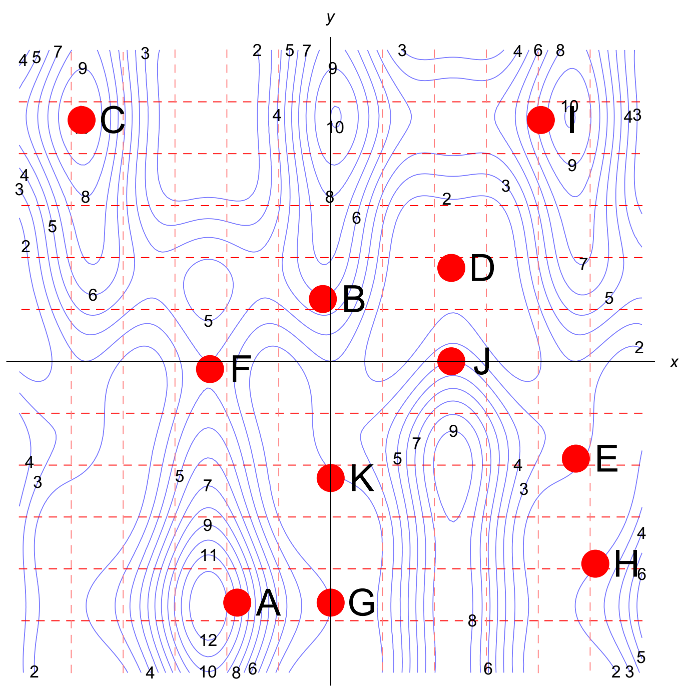


d)

Problem 3) (10 points) (No justifications are needed.)

a) (5 points) In each part, pick the correct point in $A - K$. There is an answer to each case. We don't tell you whether a point can appear twice.

	Choose one A-K
A point where $f_x = 0$, $f_{xx} < 0$ and $f_y > 0$	
A point where $f_y = 0$, $f_{yy} < 0$ and $f_x > 0$	
A local maximum	
A saddle point	
A local maximum under the constraint $x = 0$	



b) (5 points) Fill in the missing parts of the theorems or results:

i) Assume $f_x(0,0) = 0, f_y(0,0) = 0$: If $D(0,0) > 0$ and $f_{xx}(0,0) < 0$ the $(0,0)$ is a local [ ] .

ii) The gradient $\nabla f(0,0)$ is [ ] to the level curve $f(x,y) = d$ where d is the constant $d = f(0,0)$.

iii) For all continuous functions $f(x,y)$ the identity $\int_2^3 \int_5^8 f(x,y)\, dydx \int_5^8 \int_2^3 f(x,y)\, dxdy$ is assured by the [ ] theorem.

iv) The surface area of a parametrized surface $\vec{r}(u,v)$ parametrized over a region R is $\int\int_R$ [ ] $dudv$.

v) If $(0,0)$ is not a critical point of f, then the directional derivative of f in the direction $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$ is known to be [ ].

Problem 4) (10 points)

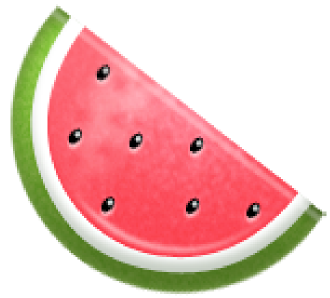
The **Emoji** with hex number 1F4C6 is a melon shaped candy. The outer radius is x, the inner is y. Assume we want to maximize the **sweetness function**

$$f(x,y) = x^2 - 2y^2$$

under the constraint that

$$g(x,y) = x - y = 2 \; .$$

Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method!

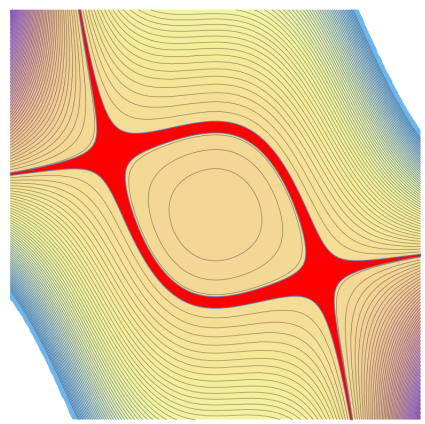


Problem 5) (10 points)

We go into the Emoji **design business**. But instead of using the Google Emoji online generator written for children ("code your own Emoji character"), we use Math and level curves. Lets take the function

$$f(x,y) = x^2 + 2x^3y + y^2 \; .$$

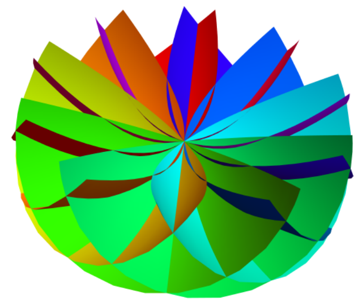
Find and classify all the critical points.



The picture shows the regions $0.23 < f(x,y) < 0.3$. We don't know what **emotion** this Emoji should represent, but it just somehow **looks cool**.

Problem 6) (10 points)

We build a **paper origami** using 10 surface leafs. The picture is seen to the right. We look at one leaf only.



Find the surface area of this surface parametrized by

$$\vec{r}(u,v) = [u+v, u-v, u^2+v^2] ,$$

with $u^2+v^2 \leq 4, v > 0$.

Problem 7) (10 points)

The **Ramanujan constant** $e^{\pi\sqrt{163}} = 262537412640768743.99999999999925...$ is close to an integer. There is an elaborate story about why this is so. Here, we just want to estimate the logarithm of this constant roughly.

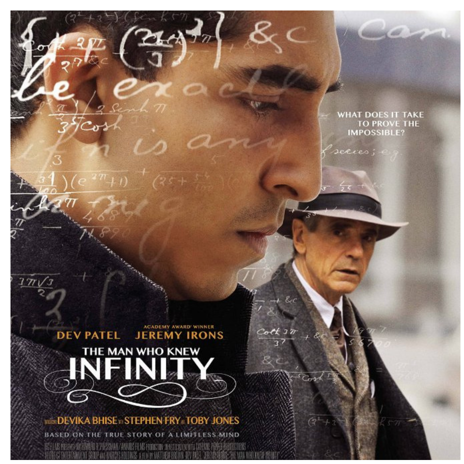
Let

$$f(x,y) = x\sqrt{y} .$$

Estimate

$$f(3.141, 163) = 3.141\sqrt{163}$$

near $(x_0, y_0) = (3, 169)$ using linear approximation.



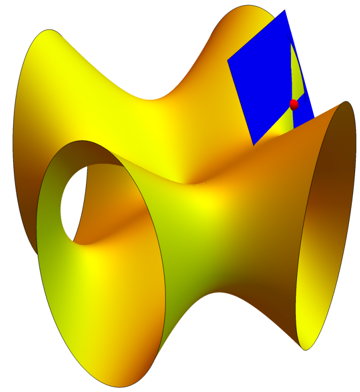
Ramanujan is featured in the movie: "The Man who knew infinity", 2015

Problem 8) (10 points)

a) (5 points) Find the tangent plane to the surface

$$x^2 + y^2 - x^2y^2 - z^2 = 0$$

at the point $(x, y, z) = (1, 2, 1)$.



b) (5 points) Find the tangent line to the curve

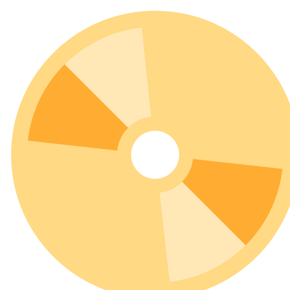
$$x^2 + y^2 - x^2y^2 = -23$$

at the point $(x, y) = (3, 2)$.

Problem 9) (10 points)

a) (5 points) Integrate

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dxdy \, .$$



b) (5 points) Integrate the double integral

$$\int \int_R (x^2 + y^2)^{5/2}\, dxdy \, ,$$

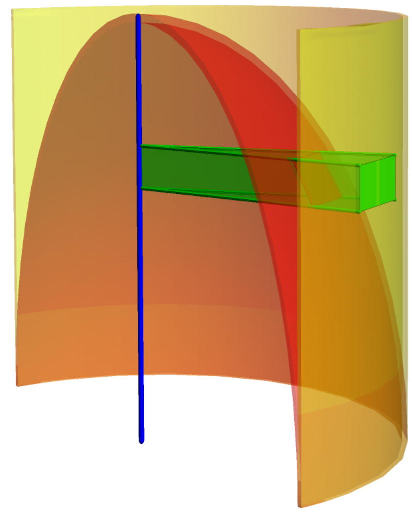
where R is the region $1 \le x^2 + y^2 \le 4$ and $x \le 0, y \ge 0$.

Problem 10) (10 points)

a) (7 points) Compute $A = |\vec{r}_\theta \times \vec{r}_\phi|$ for the half cylinder parametrized by

$$\vec{r}(\theta, \phi) = [\cos(\theta), \sin(\theta), \cos(\phi)] \ .$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ and use this to find the surface area of the half cylinder



b) (3 points) Compute $B = |\vec{r}_\theta \times \vec{r}_\phi|$ for the quarter sphere parametrized by

$$\vec{r}(\theta, \phi) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$$

with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi$ to show that (remarkably!) it is the same factor than in part a).

Remark: The fact that the surface area elements A and B are the same has been realized by Archimedes already. It allowed him to compute the surface area of the sphere in terms of the surface area of the cylinder.