7/22/2021 SECOND HOURLY PRACTICE 1 Maths 21a, O.Knill, Summer 2021

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Please email the PDF as an email attachment to knill@math.harvard.edu. The file needs to have your name capitalized like OliverKnill.pdf. Use **your personal handwriting**, no typing. No books, calculators, computers, or other electronic aids are allowed. (You can use a tablet to write). You can consult with a single page of your own handwritten notes, when writing the exam. The exam needs to arrive on Friday, July 23 at 10 AM. Write clearly and always give details of your computations. If you use separate paper, sign it with the honor code statement, use a page for each problem and copy the structure of the **check boxes**. All your final answers need to be in the boxes.

Problem 1) (20 points) No justifications are needed.

1) [T] F — If $\vec{v} = \nabla f(0,0)/|\nabla f(0,0)|$, then $D_{\vec{v}} f(0,0) = |\nabla f(0,0)|$.

Solution:

Angles dance upwards.

2) [T] F — If $\vec{v} = \vec{r}\,'(0)$ is a unit vector and $\vec{r}(0) = \vec{0}$, then $\frac{d}{dt} f(\vec{r}(t))|_{t=0} = D_{\vec{v}} f(0,0)$.

Solution:

By definition

3) [T] F — If $f(x,y,z) = 1$ defines a surface $z = g(x,y)$ with $g(1,1) = 2$ near $(x,y) = (1,1)$ then $g_x(1,1) = -f_x(1,1,2)/f_z(1,1,2)$.

Solution:

This is the implicit differentiation formula.

4) T [F] — If $D_{[1,0]} f(1,1)$ is zero, then $(1,1)$ is a critical point of $f(x,y)$.

Solution:

This is only checking $f_x = 0$.

5) T F There exists a function $f(x, y)$ for which all points $x = y$ are critical points.

Solution:

like $f(x, y) = (x - y)^2$

6) T F The function $f(x, y) = y^2$ on the constraint $g(x, y) = x = 0$ has a global minimum.

Solution:

Just put $x = 0$ and see $-3y^2$.

7) T F If $(0, 0)$ is a critical point of $f(x, y)$ satisfying $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) > 0$ and $f_{xy}(0, 0) = 0$ then f has a minimum at $(0, 0)$.

Solution:

Use the second derivative test.

8) T F The function $f(x, y) = x^2y - y$ has only one critical point.

Solution:

Look at the gradient. It is $[2xy, x^2 - 1]$ which gives $x = 1, -1, y = 0$.

9) T F If $\vec{r}(u, v)$ parametrizes a surface, then $2\vec{r}_u + 3\vec{r}_v$ is tangent to the surface.

Solution:

$\vec{r}_u$ and $\vec{r}_v$ are both tangent to the surface.

10) T F The surface area of a surface parametrized by $\vec{r}(u, v)$ over a domain R in the uv-plane is smaller or equal than $\int\int_R |\vec{r}_u \cdot \vec{r}_v| \, dudv$.

Solution:

We need $|\vec{r}_u \times \vec{r}_v|$.

11) **T** F If $f_{xx} = 0$ and $f_{xy} = 1$ for all (x, y) in the plane, then all critical points of f are saddle points.

Solution:

Yes, this implies $D < 0$.

12) **T** F For $\vec{u} = [0, 1]$ and $\vec{v} = [1, 0]$ the discriminant D in the second derivative test satisfies $D = (D_{\vec{u}}(D_{\vec{u}}f))(D_{\vec{v}}(D_{\vec{v}}f)) - (D_{\vec{u}}D_{\vec{v}}f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.

Solution:

This is the definition, if noticing that $f_u = D_u f$.

13) **T** F If $\vec{r}(t)$ parametrizes $f(x, y) = 1$, then the velocity vector $\vec{r}\,'(t)$ is perpendicular to $\nabla f(\vec{r}(t))$ for all t.

Solution:

By the gradient theorem

14) **T** F The function $f(x, y) = x^3y^2 + x^2y^3$ solves the PDE $f_{xyxyxy} = 0$.

Solution:

Use Clairaut's theorem.

15) T **F** If $f_x(x, y) = -f_y(x, y)$ for all x, y, then $f_{xx}(x, y) + f_{yy}(x, y) = 0$ for all (x, y).

Solution:

Differentiate f_x with respect to y to get f_{xy}. Differentiate f_y with respect to x to get f_{yx}. By Clairaut these are the same.

16) | T | F | (T highlighted)

The linearization of $f(x,y) = xy$ at $(3,4)$ is $L(x,y) = 12+4(x-3)+3(y-4)$.

Solution:

The

17) | T | F | (T highlighted)

If $f(x,t)$ solves the partial differential equation $f_t = f_{xx}$, then $g(x,t) = f(t,x)$ solves the partial differential equation $g_x = g_{tt}$.

Solution:

You differentiate twice

18) | T | F | (F highlighted)

There is a function f such that $D_{\vec{u}} f(0,0)$ is negative for all unit vectors $\vec{u}$.

Solution:

If the function is positive in one direction, then it is negative in the opposite direction.

19) | T | F | (T highlighted)

The surface area of the unit sphere does not depend on the parametrization.

Solution:

It is a general fact for parametrizations

20) | T | F | (T highlighted)

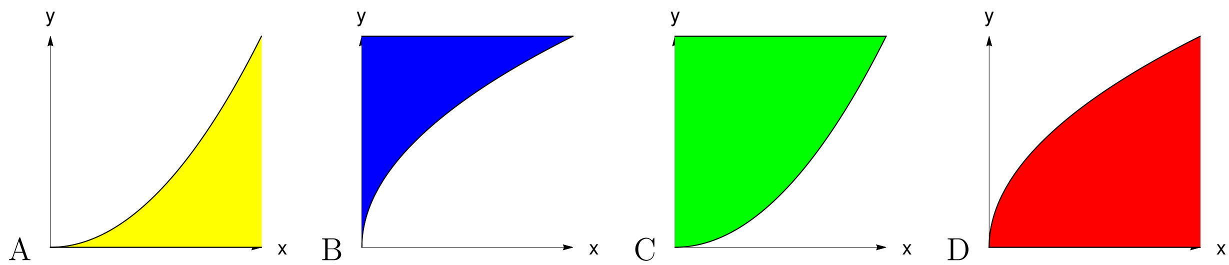
There is an f with a critical point with discriminant $D = 0$ and $f_{xx} > 0$ and $f_{xy} = 0$.

Solution:

$f = x^2 + y^4$ is possible.

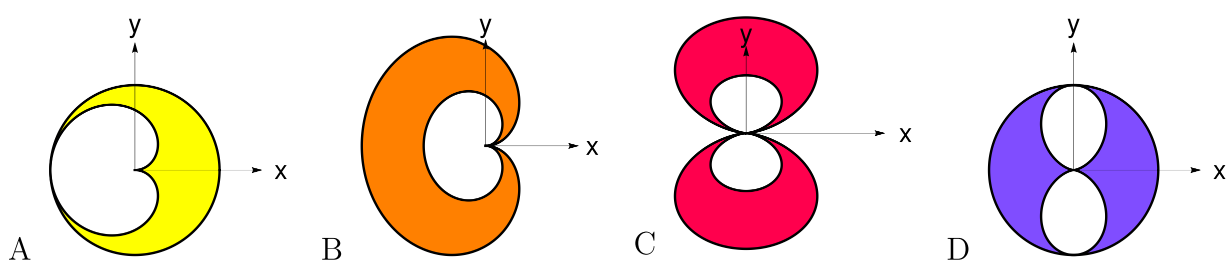
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with their area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{y^2}^1 1 \, dxdy$
	$\int_0^1 \int_0^{x^2} 1 \, dydx$
	$\int_0^1 \int_0^{\sqrt{y}} 1 \, dxdy$
	$\int_0^1 \int_{\sqrt{x}}^1 1 \, dydx$

b) (4 points) Match the regions with their area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^1 r \, dr d\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^1 r \, dr d\theta$
	$\int_0^{2\pi} \int_{\sin(\theta/2)}^{2\sin(\theta/2)} r \, dr d\theta$
	$\int_0^{2\pi} \int_{\sin^2(\theta)}^{2\sin^2(\theta)} r \, dr d\theta$

c) (2 points) Name all the four PDE's for a function $\psi(B, C)$ of variables B and C.

$\psi_B = \psi_C$	$\psi_{BB} + \psi_{CC} = 0$	$\psi_B + \psi\psi_C = \psi_{CC}$	$\psi_B = \psi - C\psi_C - C^2\psi_{CC}$

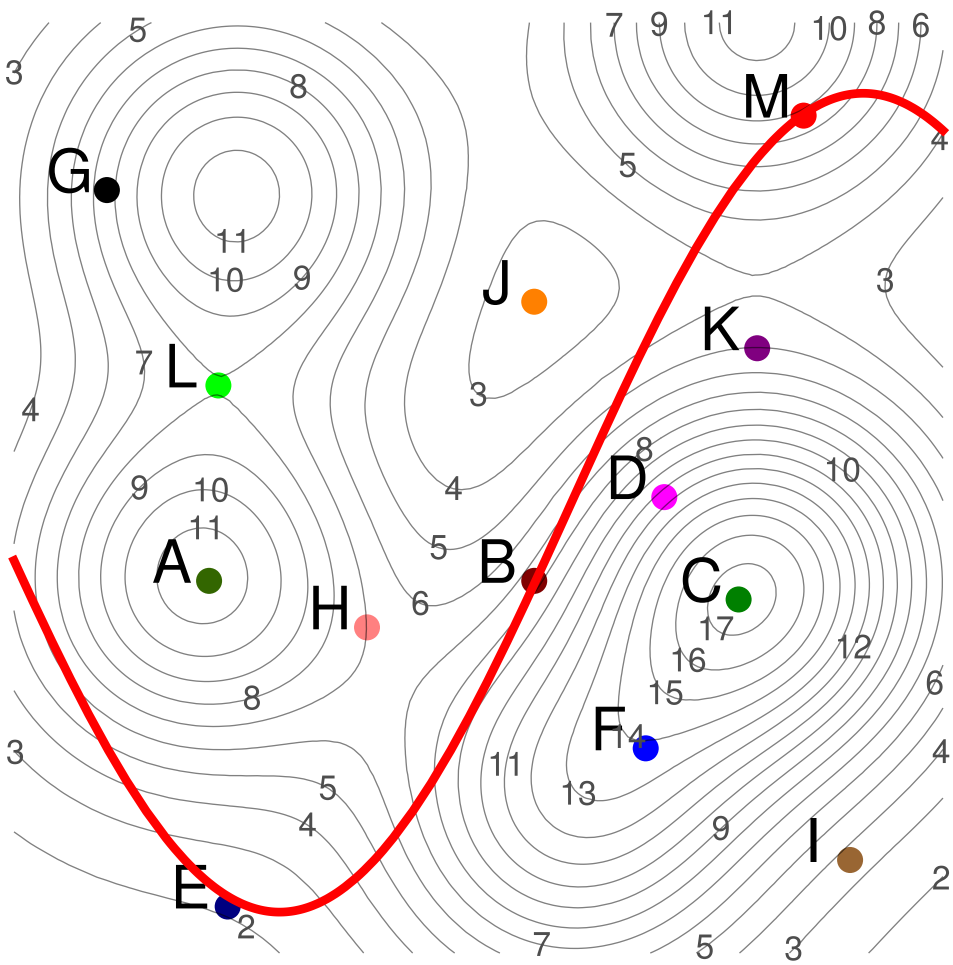
Solution:

a) DACB
b) DABC
c) Transport, Laplace, Burgers, Black Sholes

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We see the contours of an unknown smooth function, $f(x, y)$. The thick red curve is $g(x, y) = y - \sin(x) = 1$. Use each of the labels A-M only once. There are three labels which do not match.

	Enter A-M
A point, where $f_x > 0, f_y = 0$.	
A point, where $f_y > 0, f_x = 0$.	
A point, where $f_y < 0, f_x = 0$.	
A saddle point of f.	
A local minimum of $f(x, y)$.	
A local but not global maximum of $f(x, y)$.	
A global maximum of $f(x, y)$.	
The point among A-M with maximal $\|\nabla f\|$.	
A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$.	
A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$.	



Solution:

G,F,K,L,J,A,C,D,M,E.

Problem 4) (10 points)

a) (8 points) Classify the critical points of the function

$$f(x,y) = 4x^2 - 4y^2 - 2x^4 + 8y$$

using the second derivative test.

Point	D	f_{xx}	nature

b) (2 points) Is there a global maximumn or minimum of $f(x,y)$? (No explanation necessary for this part b).)

	Yes	No
There is a global max for f		
There is a global min for f		

Solution:

a)

x	y	D	$f_x x$	Type	f
-1	1	128	-16	maximum	6
0	1	-64	8	saddle	4
1	1	128	-16	maximum	6

b) Global Max but no global min.

Problem 5) (10 points)

Find the minimum of the function

$$f(x,y) = 5 + x^2 + y^2 + 2xy$$

under the constraint $x + 2y = 5$. You need to use the Lagrange method.

$(x,y) =$ [ ]

Solution:

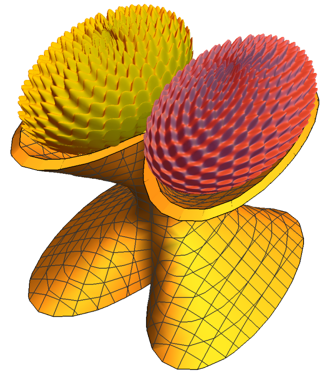
The Lagrange equations lead to $x = -y$. The solution is $(-5, 5)$

Problem 6) (10 points)

a) 5 points) Find the equation $ax + by + cz = d$ of the tangent plane to the surface

$$f(x, y, z) = x^4 + y^4 + z^2 + x^2y^2 - x^2z^2 + y^2 - z^2 = 3$$

at the point $(1, 1, 1)$.



Ice cream sponsored by math-candy.com.

Plane

b) (5 points) Estimate $f(1.01, 1.02, 0.97)$ using linearization.

Estimate

Solution:

a) $2x + 4y - z = 5$.
b) $3 + 4 * 0.01 + 8 * -.02 + (-2) * (-0.03) = 3.26$.

Problem 7) (10 points)

Find the surface area of the surface

$$\vec{r}(u, v) = [\frac{v^2}{2} - 3, \frac{u^2}{2} + 3, \frac{uv}{\sqrt{2}}] .$$

for which the parameters satisfy $u^2 + v^2 \leq 4$.

Surface area

Solution:

$$|\vec{r}_u \times \vec{r}_v| = (u^2 + v^2)/\sqrt{2}$$

To integrate over the parameter region, we use polar coordinates:

$$\int_0^{2\pi} \int_0^2 r^2/\sqrt{2}\ r\ dr d\theta\ = 4\sqrt{2}\pi = 8\pi/\sqrt{2}\ .$$

Problem 8) (10 points)

Assume we know $f_x(3,3) = 1$ and $D_{[1,1]/\sqrt{2}} f(3,3) = 2\sqrt{2}$.

a) (5 points) Find the tangent line to the curve $\{f(x,y) = 7\}$ at $(3,3)$.

Tangent line

b) (5 points) Estimate $f(3.0001, 3.003)$ using linearization.

Estimate

Solution:

a) $f_x = 1, f_y = 3,\ x + 3y = 12$
b) $7 + 0.0001 + 0.009 = 7.0091.$

Problem 9) (10 points)

a) (5 points) Evaluate the following double integral

$$\int\int_G (x^2 + y^2)^4 \; dxdy \; ,$$

where G is region given by

$$\{x^2 + y^2 \leq 4, x \geq 0, y \geq 0\} \; .$$

Result

b) (5 points)

$$\int_0^1 \int_0^{\arctan(x)} \frac{1}{1 - \tan(y)} \; dydx \; .$$

Result

Solution:

a) $\int_{\pi/2} \int_0^2 r^8 \cdot r \; dr d\theta = 2^{10}\pi/20 = 256\pi/5.$
b) Switch the order of integration. To do so, make a picture.

$$\int_0^{\pi/4} \int_1^{\tan(y)} \frac{1}{1 - \tan(y)} dxdy = \int_0^{\pi/4} 1 \; dy = \frac{\pi}{4} \; .$$