7/8/2021 FIRST HOURLY Practice 1 **Maths 21a, O.Knill, Summer 2021**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Please email the PDF as an email attachment to knill@math.harvard.edu. The file needs to have your name capitalized like OliverKnill.pdf. Use **your personal handwriting**, no typing. No books, calculators, computers, or other electronic aids are allowed. (You can use a tablet to write). You can consult with a single page of your own handwritten notes, when writing the exam. The exam needs to arrive on Friday, July 9 at 10 AM. Write clearly and always give details of your computations. If you use separate paper, sign it with the honor code statement, use a page for each problem and copy the structure of the **check boxes**. All your final answers need to be in the boxes.

Problem 1) (20 points) No justifications are needed.

1) T F The plane $x = 3$ does intersect the yz-plane.

Solution:

The xy plane is $x = 0$.

2) T F The curve $\vec{r}(t) = [1 + 2t, t, 1 + t]$ intersects the z-axis in a point.

Solution:

If $y = 0$, then x is not zero.

3) T F The Cauchy-Schwartz inequality states $|\vec{v} \cdot \vec{w}| \leq |\vec{v}|$ for any two vectors $\vec{v}, \vec{w}$.

Solution:

We also have to divide by the length of the vector $\vec{v}$.

4) T F The curvature of a circle $\vec{r}(t) = [\cos(2t), 0, \sin(2t)]$ is equal to 1/2 everywhere.

Solution:

This is a unit circle, the curvature is 1.

5) **T** F

The surface $y^2 - x + y^2 = 2$ is an elliptic paraboloid.

Solution:

It is an elliptic paraboloid.

6) T **F**

The angle between the vectors $\vec{0A}$ and $\vec{0B}$ is positive if the distance between the points A and B is positive.

Solution:

The angle can become zero.

7) **T** F

Let $\vec{j} = [0, 1, 0]$. There is a vector $\vec{v}$ for which the vector projection of $\vec{v}$ onto $\vec{j}$ is equal to $-\vec{j}$.

Solution:

Take $\vec{v} = -\vec{j}$.

8) T **F**

Two particles with path $\vec{r}_1(t) = [0, t, -t]$ and $\vec{r}_2(t) = [1 - t, t - 1, 0]$ do collide.

Solution:

They do intersect as curves but not collide.

9) **T** F

In spherical coordinates the surface $\rho^2 \sin^2(\phi) - \rho^2 \cos^2(\phi) = 1$ is a one-sheeted hyperboloid.

Solution:

It is $x^2 + y^2 - z^2 = 1$.

10) **T** F

If $|\vec{u} \times \vec{v}| = 1$, for unit vectors $\vec{u}, \vec{v}$, then $\vec{u}, \vec{v}$ are orthogonal.

Solution:

We must have $\sin(\alpha) = 1$.

11) [T] F — The curve $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1$ in polar coordinates is a hyperbola.

Solution:

Yes, $x^2 - y^2 = 1$.

12) [T] F — If the arc length of a curve connecting A with B is 0, then $A = B$.

Solution:

The arc length is larger or equal than the distance.

13) [T] F — The surface parametrized as $\vec{r}(y, z) = [y, z, y^2 - z^2]$ is a hyperbolic paraboloid.

Solution:

Yes, it reads $z = x^2 + y^2$.

14) T [F] — The velocity vector and the acceleration are always either parallel or perpendicular.

Solution:

These are extreme cases but it is not true in general

15) [T] F — It is possible that a plane and a one-sheeted hyperboloid intersects in two crossing lines.

Solution:

See homework

16) T F The function $f(x,y) = \log(x^2 + y^2)$ contains as domain all points except the origin $(0,0)$.

Solution:

Yes, the log

17) T F The normal vector $\vec{N}$ and the unit tangent vector $\vec{T}$ are perpendicular if $\vec{T}, \vec{T}\,'$ are both not zero vectors.

Solution:

We have proven this

18) T F The distance between two non-parallel lines in three dimensional space can be zero.

Solution:

Yes, If they cross.

19) T F If $\vec{i}, \vec{j}, \vec{k}$ denote the unit vectors in the x, y and z axis, then $\vec{i} \cdot (\vec{j} \times \vec{k}) = 1$.

Solution:

Yes it is the volume.

20) T F For any two lines L, M, there are points P on L and Q on M such that $d(P,Q) = 2d(L,M)$, where $d(L,M)$ is the distance between the lines.

Solution:

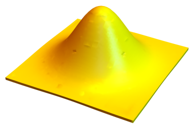
By the intermediate value theorem

Problem 2) (10 points) No justifications are needed in this problem.

In each sub-problem, each of the numbers 0,1,2,3 each occur exactly once.

a) (2 points) Match the surfaces $g(x,y,z) = c$. Enter 0 if there is no match.

1



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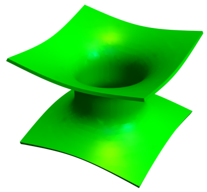

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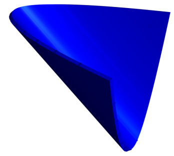
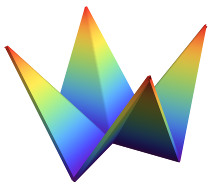


Function $g(x,y,z) =$	0,1,2, or 3
$x - y^2 + z = 0$	
$x^2 + y^2 - z^4 = 1$	
$z - e^{-x^2-y^2} = 0$	
$y^2 + z^3 = 1$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter 0 if there is no match.

1



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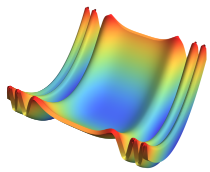

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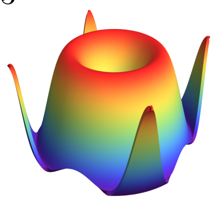
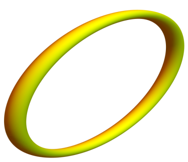


Function $f(x,y) =$	0,1,2, or 3
$\sqrt{\|1 + x^2 - y^2\|}$	
$\sin(x^2 + y^2)$	
$\|\|x - y\| - \|x + y\|\|$	
$y^2 \sin(x^4)$	

c) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

1



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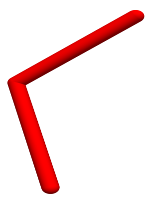

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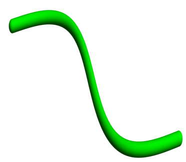
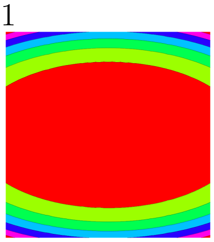


Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[\|t\|, t, t]$	
$[\cos(2t), 0, \cos(2t)]$	
$[0, \cos(2t), \sin(2t)]$	
$[t, \sin(t), 0]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter 0 if there is no match.

1



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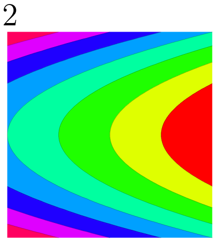

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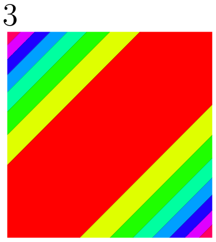
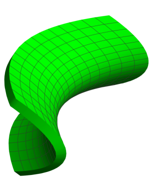


Function $g(x,y) =$	0,1,2, or 3
$\cos(2x) + \sin(2y)$	
$(x - y)^2$	
$y^2 - x$	
$(2x^2 + 7y^2)^2$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1



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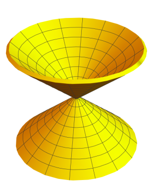

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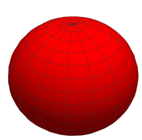


Parametrization $\vec{r}(u,v) =$	0-3
$[u, u^2 + v^2, v]$	
$[\sin(v)\cos(u), \sin(v)\sin(u), \cos(v)]$	
$[u^2 - v^2, u, v]$	
$[u\cos(v), u\sin(v), u]$	

Solution:

a) 3,2,1,0
b) 0,3,1,2
c) 2,0,1,3
d) 0,3,2,1
e) 0,3,1,2

Problem 3) (10 points)

Let $\vec{v} = [2, 2, 1]$ and $\vec{w} = [1, 1, 1]$ and assume θ is the angle between $\vec{v}$ and $\vec{w}$.

a) (5 points) Compute $\vec{n} = \vec{v} \times \vec{w}$ and $a = \vec{v} \cdot \vec{w}$.

$\vec{n} =$ [ ] $a =$ [ ]

b) (5 points) Compute $\sin^2(\theta)$ by using the vector $\vec{n}$ and $\cos^2(\theta)$ by using the scalar a and check that $\sin^2(\theta) + \cos^2(\theta)$ is equal to 1.

$\sin^2(\theta) =$ $\cos^2(\theta) =$

Check that the sum is 1:

Solution:

a) $[1, -1]$ and 5. b) $2/27$ and $25/27$. Check: $2/27 + 25/27 = 1$.

Problem 4) (10 points)

a) (5 points) Parametrize the plane $4x + 4y + 2z = 0$ using parameters s, t:

$\vec{r}(s, t) =$

b) (5 points) Find the distance between $4x + 4y + 2z = 0$ and the point $(2, 2, 2)$.

distance =

Solution:

a) possibility: $[x, y, z] = s[1, -1, 0] + t[1, 0, -2]$. b) Use the distance formula: 10/3.

Problem 5) (10 points)

a) (5 points) Find a parametrization $\vec{r}(t)$ of the intersection of the planes

$$x + y + z = 1, \qquad 2x - y + 2z = 2 .$$

$\vec{r}(t) =$

b) (5 points) Find the distance between that line computed in a) and $P = (1, 1, 1)$.

distance =

Solution:

a) Possibility: $[1,0,0] + [t,0,-t]$. The easiest was to find two points on the intersection like $A = (1,0,0)$ and $B = (2,0,-1)$.
b) Distance formula $\sqrt{3/2} = \sqrt{6}/2$.

Problem 6) (10 points)

a) (5 points) Find the arc length of the path

$$\vec{r}(t) = [\frac{3t^2}{2}, \frac{4t^2}{2}, \frac{5t^3}{3}]$$

with $0 \leq t \leq 1$.

Length =

b) (5 points) Find the curvature of $\vec{r}(t)$ at $t = 1$ using the vectors $\vec{v} = \vec{r}\,'(1), \vec{w} = \vec{r}\,''(1)$.

$\kappa(\vec{r}(1))) =$

Solution:

a) We have to integrate $\sqrt{25t^2 + 25t^4}$ from $t = 0$ to $t = 1$. Note that we can factor a t outside the square root, then use substitution and get $(5/3)(2^{3/2} - 1) = (10\sqrt{2} - 5)/3$.
b) $1/(2\sqrt{50}) = \sqrt{2}/20 = 1/(10\sqrt{2})$.

Problem 7) (10 points)

a) (5 points) Given the **jerk**

$$\vec{r}\,'''(t) = [0, 0, -12]$$

with $\vec{r}(0) = [0, 0, 3], \vec{r}\,'(0) = [4, 0, 0], \vec{r}\,''(0) = [0, 2, 0]$, find $\vec{r}(t)$ and especially $\vec{r}(10)$.

$\vec{r}(10) =$

b) (5 points) Compute the unit tangent vector $\vec{T}$ of the TNB-frame to the curve $\vec{r}(t)$ at $t = 0$.

$\vec{T}(0) =$

Solution:

a) Just integrate $r''(t) = [0, 2, -12t]$, $r'(t) = [3, 2t, -6t^2]$, $r(t) = [3t, t^2, -3t^3]$. The point is $(40, 100, -1997)$.
b) The velocity was already given. Just normalize $[1, 0, 0]$.

Problem 8) (10 points)

We experiment with a **paper air plane**. The wing tips are $C = (0, 3, 4)$ and $D = (0, -3, 4)$ the front is $A = (3, 0, 0)$ the back is $B = (-3, 0, 0)$.

a) (5 points) The sum of the areas of the triangles ABC and ABD is the total wing area. Find the wing area.

Area =

b) (5 points) The volume of the tetrahedron with vertices A, B, C, D is known to be 1/6th of the volume of the parallelepiped spanned by AB, AC, AD. Find the volume of the tetrahedron.

Volume = [ ]

Solution:

a) Crossproduct/2 in each triangle gives 15. The total area is $15 + 15 = 30$.
b) Triple scalar product is 24.

Problem 9) (10 points) No justifications are needed.

a) (2 points) Parametrize the surface $x^2/9 + y^2/4 + z^2/4 = 1$.

$\vec{r}(\theta, \phi) = [\quad , \quad , \quad]$

b) (2 points) Parametrize the surface $x^2 = y^2 + z^2$ using an angle θ in the yz-plane.

$\vec{r}(\theta, x) = [\quad , \quad , \quad]$

c) (2 points) Parametrize the surface $y = x^4 - 6x^2z^2 + z^4$.

$\vec{r}(x, z) = [\quad , \quad , \quad]$

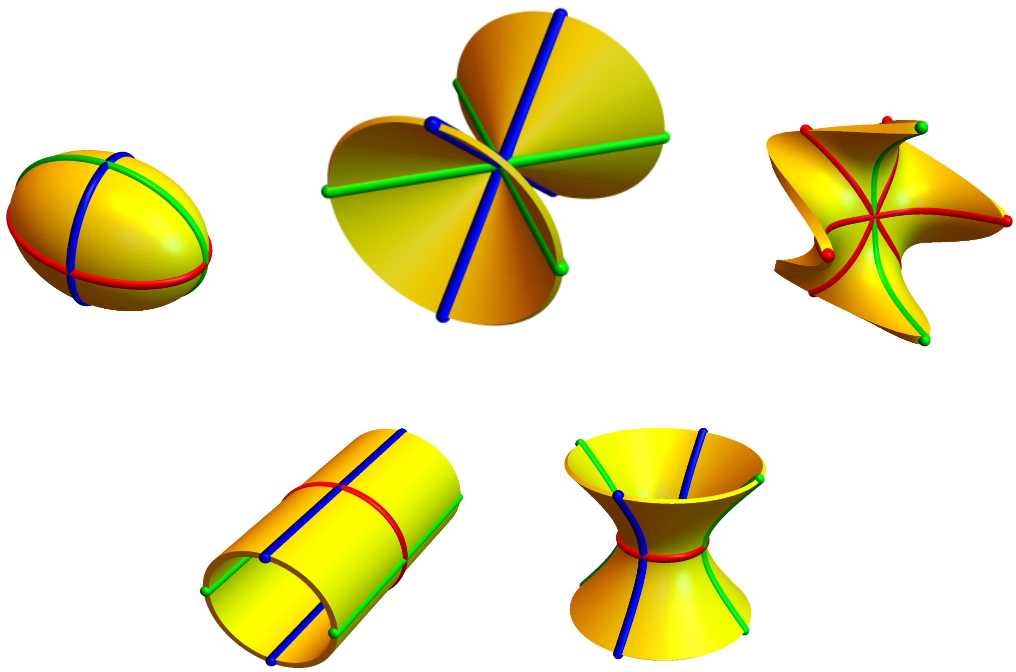
d) (2 points) Parametrize the surface $x^2 + z^2 = 4$ using an angle θ in the xz-plane.

$\vec{r}(\theta, y) = [\quad , \quad , \quad]$

e) (2 points) Parametrize the surface $(x - 7)^2 + z^2 - y^2 = 8$.

$\vec{r}(\theta, y) = [\quad , \quad , \quad]$

The following pictures are here just to be admired and can also be ignored.



Solution:

a) $[3\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)]$.
b) $[x, x\cos(\theta), x\sin(\theta)]$.
c) $[x, x^4 - 6x^2z^2 + z^4, z]$.
d) $[2\cos(\theta), y, 2\sin(\theta)]$.
e) $[7 + \sqrt{8 + y^2}\cos(\theta), y, \sqrt{8 + y^2}\sin(\theta)]$.