7/8/2021 FIRST HOURLY **Maths 21a, O.Knill, Summer 2021**

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Please email the PDF as an email attachment to knill@math.harvard.edu. The file needs to have your name capitalized like OliverKnill.pdf. Use **your personal hand-writing**, no typing. No books, calculators, computers, or other electronic aids are allowed. (You can use a tablet to write). You can consult with a single page of your own handwritten notes, when writing the exam. The exam needs to arrive on Friday, July 9 at 10 AM. Write clearly and always give details of your computations. If you use your own paper, sign it with the honor code statement, use a page for each problem and copy the structure of the **check boxes**. All your final answers need to be in the boxes.

Problem 1) (20 points) No justifications are needed.

1) T **F** For any vector $\vec{v} = [a, b, c]$ we have $|[a, b, c]| = [|a|, |b|, |c|]$.

Solution:

We used the absolute sign for vectors as magnitude.

2) **T** F The curvature of a curve at a point is independent of the parametrization.

Solution:

A basic property.

3) T **F** It is possible to intersect a cylinder with a plane and get a hyperbola.

Solution:

The intersection is either empty, a line or then an ellipse.

4) **T** F If $\vec{T}, \vec{N}, \vec{B}$ is a TNB frame then $\vec{N} = \vec{B} \times \vec{T}$

Solution:

The three vectors are an orthonormal frame like $\vec{T} = [1, 0, 0], \vec{N} = [0, 1, 0], \vec{B} = [0, 0, 1]$.

5) T F

The intersection between two spheres of radius 1 and 2 is either empty, a point, a circle.

Solution:

Just draw all the possible situations.

6) T F

The set of points in space which satisfy $x^2 - y^2 = 1$ form a hyperbola.

Solution:

It is a cylindrical hyperboloid.

7) T F

The length of the sum of two vectors in space is always larger or equal than the sum of the lengths of the vectors.

Solution:

Just opposite

8) T F

For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the identity $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + \vec{u} \times \vec{w}$ holds.

Solution:

One calls this distributivity.

9) T F

The set of points which satisfy $-x^2 - 2x + y^2 + z^2 = 1$ define a cone.

Solution:

Complete the square.

10) T F

If A, B, C are three points space which are not contained in a common line, then $\vec{AB} \times \vec{AC}$ is a vector orthogonal to the plane containing A, B, C.

Solution:

The condition assures that the vectors $\vec{AB}$ and $\vec{AC}$ are not parallel.

11) T F The line $\vec{r}(t) = [t, 5t, 4t]$ hits the plane $x + 5y + 4z = 100$ at a right angle.

Solution:

Indeed, the normal vector $[1, 5, 4]$ is also the velocity vector of the curve.

12) T F The surface given in (r, θ, z) coordinates as $r = \sin(\theta)$ is a paraboloid.

Solution:

It translates to $r^2 = r\sin(\theta)$ which is $r^2 = r$ meaning $r = 1$ or $r = 0$.

13) T F If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$, then $\vec{v}$ and $\vec{w}$ are parallel (in the sense that there exists a constant c such that $\vec{v} = c\vec{w}$).

Solution:

The condition means that the cross product is zero implying parallel vectors.

14) T F If $|\vec{x} \times \vec{v}| = 0$ for all vectors $\vec{v}$, then $\vec{x} = \vec{0}$.

Solution:

That means that $\vec{x}$ is perpendicular to all vectors. This is only possible for the zero vector.

15) T F If $\vec{u}$ and $\vec{v}$ are orthogonal, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

Solution:

$\vec{u}, \vec{v}, \vec{w} = \vec{u} \times \vec{v}$ are all perpendicular. The vector under consideration is perpendicular both to $\vec{w}$ and to $\vec{u}$ and so parallel to $\vec{v}$.

16) T **F** Every vector contained in the line $\vec{r}(t) = [4+2t, 2+3t, 3+4t]$ is parallel to the vector $(4, 2, 3)$.

Solution:

It is parallel to the vector $[2, 3, 4]$.

17) **T** F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1/2, 3\pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, -1/2, 0)$.

Solution:

The angle conditions means to be on the negative y-axes. Now take distance $1/2$.

18) **T** F The set of points which satisfy $x^2 - 4x + 2y^2 + 3z^2 = -3$ is an ellipsoid.

Solution:

Complete the square again.

19) **T** F If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then all three vectors $\vec{u}, \vec{v}, \vec{w}$ are in the same plane.

Solution:

Indeed the volume of the parallel epiped is zero.

20) **T** F The set of points in $\mathbb{R}^3$ which have distance 1 from the curve $\vec{r}(t) = [3\cos(t), 3\sin(t), 0]$ form a torus (doughnut)

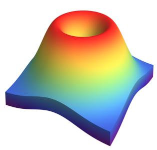
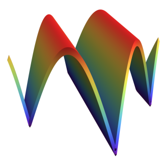
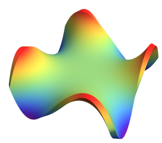
Solution:

Yes, that is a way how you can draw a doughnut. Take a circle away from the z axes and spin it around the z axes.

Problem 2) (10 points) No justifications are needed in this problem.

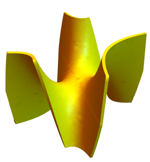
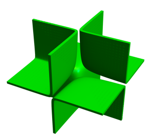
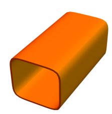
In each sub-problem, each of the numbers 0,1,2,3 each occur exactly once.

a) (2 points) Match the graphs of the functions $f(x, y)$. Enter 0 if there is no match.


1

2

3

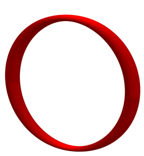
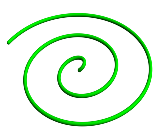
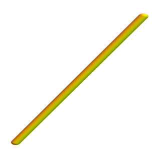
Function $f(x, y) =$	0,1,2, or 3
$xy(x^2 - y^2)$	
$e^{-x^2-y^2}(x^2 + y^2)$	
$1/(x^2 + y^4 + 1)$	
$\lvert \sin(x + y)\rvert$	

b) (2 points) Match the surfaces $g(x, y, z) = c$. Enter 0 if there is no match.


1

2

3

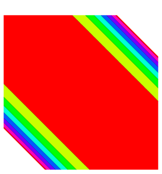
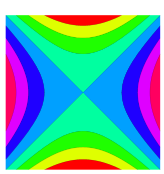
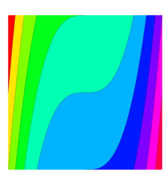
Function $g(x, y, z) =$	0,1,2, or 3
$1000 * \sin(xyz) = 1$	
$x^6 + z^6 = 1$	
$z - (x^2 - y^2)x = 0$	
$x - y^2 = 1$	

c) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.


1

2

3

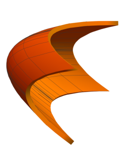
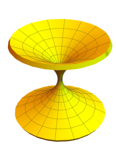
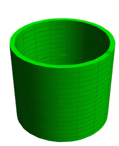
Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[t\cos(t), t\sin(t), \sin(t)]$	
$[\cos(3t), 0, 3\sin(3t)]$	
$[\exp(t), 2\exp(t), 3\exp(t)]$	
$[t, 0, t\sin(t)]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter 0 if there is no match.


1

2

3

Function $g(x, y) =$	0,1,2, or 3
$x^2 - y^2$	
$(x + y)^4$	
$x^3 - y$	
$\cos(3x) + \sin(3y)$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.


1

2

3

Parametrization $\vec{r}(u, v) =$	0-3
$[\cos(v), \sin(v), u]$	
$[u^2\cos(v), u^2\sin(v), u]$	
$[\sin(v)\cos(u), \cos(v), 2\sin(v)\sin(u)]$	
$[u^6 + v^6, u^3, v^3]$	

Solution:

a) 3102
b) 2310
c) 2130
d) 2130
e) 3201

Problem 3) (10 points)

We perform some computations with the vectors $\vec{v} = [2, 2, 3]$ and $\vec{w} = [1, 2, 2]$.

a) (2 points) Find the cross product $\vec{v} \times \vec{w}$?

b) (2 points) Construct a unit vector in the same direction than $\vec{v} \times \vec{w}$.

c) (2 points) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

d) (2 points) What is the vector projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

e) (2 points) Compute $(\vec{v} + \vec{w}) \cdot (\vec{v} \times \vec{w})$.

Solution:

a) $[-2, -1, 2]$
b) $[-2, -1, 2]/3$.
c) $12/(3\sqrt{17})$.
d) $12/9[1, 2, 2]$.
e) 0.

Problem 4) (10 points)

a) (2 points) Parametrize the plane containing the points $(1,0,0),(0,5,0),(0,0,2)$ using parameters s,t:

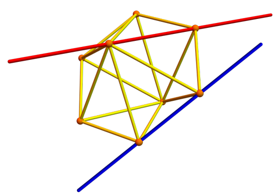
b) (3 points) Now find the equation $ax+by+cz=d$ of that plane in a).
c) (5 points) Finally find the distance between that plane defined in a) and the point $P=(3,3,3)$.

Solution:

a) A possibility is $\vec{r}(s,t)=[1-t-s,5t,2s]$.
b) $10x+2y+4z=10$.
c) $41/\sqrt{129}$.

Problem 5) (10 points)

A **prismatic polyedron** with 8 vertices contains the 4 points $A = (2, 0, 0), B = (1, \sqrt{3}, 0)$ and $C = (-1, \sqrt{3}, 0), D = (0, 0, \sqrt{3})$. Find the distance of the line containing the points A, B and the line containing the points C, D.



Solution:

This is a standard distance line-line problem: we have $\vec{AB} = [-1, \sqrt{3}, 0]$, $\vec{CD} = [1, -\sqrt{3}, \sqrt{3}]$, $\vec{BC} = [2, 0, 0]$. We have $\vec{AB} \times \vec{CD} = [3, Sqrt[3], 0]$ with $|\vec{AB} \times \vec{CD}| = \sqrt{12}$ The distance is $d = \vec{BC} \cdot (\vec{AB} \times \vec{CD})/|\vec{AB} \times \vec{CD}| = 6/\sqrt{12}$. This simplifies to $\sqrt{3}$.

Problem 6) (10 points)

a) (5 points) Find the arc length of the path

$$\vec{r}(t) = [\frac{t^3}{3}, t^2, 2t]$$

with $-1 \le t \le 1$.

b) (5 points) Compute $\vec{v} = \vec{r}\,'(1), \vec{w} = \vec{r}\,''(1)$ and express the curvature of the curve at $\vec{r}(1)$ in terms of $\vec{v}$ and $\vec{w}$.

Solution:

a) We have $\vec{r}\ '(t) = [t^2, 2t, 2]$. The speed is $|\vec{r}\ '(t) = 2 + t^2$. The integral is 14/3.
b) $\vec{v} = [1, 2, 2]$
$\vec{w} = [2, 2, 0]$
$\kappa = |\vec{v} \times \vec{w}|/|vecv|^3 = |[-4, 4, -2]|/3^3 = 6/27 = 2/9$.

Problem 7) (10 points)

On Mars, the sum of gravitational acceleration and wind force is $\vec{r}\,''(t) = [0, \sin(t), -4]$. The **Mars helicopter Ingenuity** has been rising up to 5 meters. There had been a small stone pebble stuck on one of the legs. It falls down from the position $\vec{r}(0) = [2, 1, 5]$ with velocity $\vec{r}\,'(0) = [1, 0, 0]$ subject to the acceleration given above.

a) (6 points) Determine the path of the pebble.

b) (4 points) At which time does the pebble hit the ground?

Solution:

a) Integrate and fix the constant to get $\vec{r}\,'(t) = [1, 1 - \cos(t), -4t]$.
Integrate and again and fix the constant $\vec{r}(t) = [t + 2, t - \sin(t) + 1, 5 - 2t^2]$.
b) We need $5 - 2t^2 = 0$ means $t = \sqrt{5/2}$.

Problem 8) (10 points)

We denote with $|ABC|$ the **area** of a triangle ABC defined by three points A, B, C. Take $A = (2,0,0), B = (0,3,0), C = (0,0,1)$ as well the point $O = (0,0,0)$. Verify in this case the **3D Pythagoras theorem**

$$|ABC|^2 = |ABO|^2 + |BCO|^2 + |ACO|^2 \ .$$

$|ABC|^2 =$

$\lvert ABO\rvert^2 =$	$\lvert BCO\rvert^2 =$	$\lvert ACO\rvert^2 =$

Solution:

We have $|ABC| = |\vec{AB} \times \vec{AC}|/2 = 7/2$. This could not be short cut. We also have $|ABO| = |\vec{OA} \times OB| = 3$. etc. (The three triangle areas square $|ABO|^2 = 3^2$, $|BCO|^2 = 9/4$ and $|ACO|^2 = 1$ could also be computed directly.)

Problem 9) (10 points) No justifications are needed.

a) (2 points) Parametrize the surface $3x + 2y + 4z = 12$.

$\vec{r}(s,t) = [\quad , \quad , \quad]$

b) (2 points) Parametrize the surface $y^2 + (z-1)^2 = 9$ using an angle θ in the yz-plane.

$\vec{r}(\theta,x) = [\quad , \quad , \quad]$

c) (2 points) Parametrize the surface $x^2 + 2x + y^2 + z^2/9 = 0$.

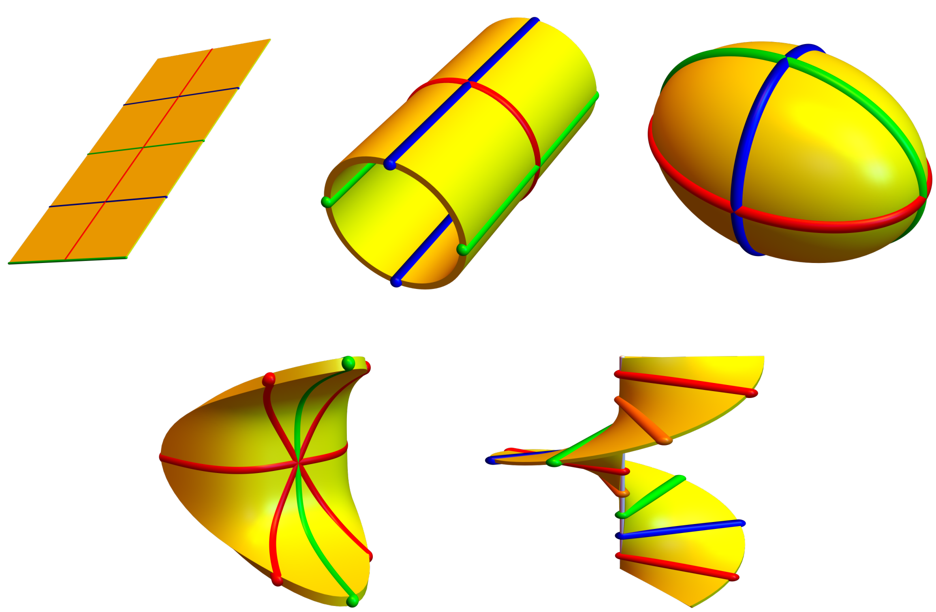
$\vec{r}(\theta,\phi) = [\quad , \quad , \quad]$

d) (2 points) Parametrize the surface $y = x^4 - z^4$.

$\vec{r}(x,z) = [\quad , \quad , \quad]$

e) (2 points) Parametrize the surface obtained by taking the helix $r(\vec{t}) = [\cos(t), \sin(t), t]$ and connect each point $\vec{r}(t)$ with the projection onto the z-axes.

$\vec{r}(t,s) = [\quad , \quad , \quad]$



Solution:

Example solutions: a) $[s, t, (12 - 3s - 2t)/4]$
b) $[x, 3\cos(t), 1 + 3\sin(t)]$
c) $[\sin(\phi)\cos(\theta) - 1, \sin(\phi)\sin(\theta), 3\cos(\phi)]$
d) $[x, x^4 - z^4, z]$
e) $[s\cos(t), s\sin(t), t]$