"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

Please email the PDF as an email attachment to knill@math.harvard.edu. The file needs to have your name capitalized like OliverKnill.pdf. Use **your personal hand-writing**, no typing. No books, calculators, computers, or other electronic aids are allowed. (You can use a tablet to write). You can consult with a single page of your own handwritten notes, when writing the exam. The exam needs to arrive on Friday, July 9 at 10 AM. Write clearly and always give details of your computations. If you use your own paper, sign it with the honor code statement, use a page for each problem and copy the structure of the **check boxes**. All your final answers need to be in the boxes.
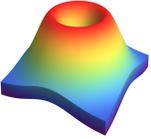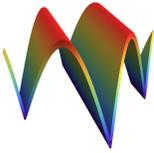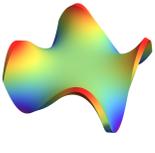
Problem 1) (20 points) No justifications are needed.

1)   T   F   For any vector $\vec{v} = [a, b, c]$ we have $|[a, b, c]| = [|a|, |b|, |c|]$.

2)   T   F   The curvature of a curve at a point is independent of the parametrization.

3)   T   F   It is possible to intersect a cylinder with a plane and get a hyperbola.

4)   T   F   If $\vec{T}, \vec{N}, \vec{B}$ is a TNB frame then $\vec{N} = \vec{B} \times \vec{T}$

5)   T   F   The intersection between two spheres of radius 1 and 2 is either empty, a point, a circle.

6)   T   F   The set of points in space which satisfy $x^2 - y^2 = 1$ form a hyperbola.

7)   T   F   The length of the sum of two vectors in space is always larger or equal than the sum of the lengths of the vectors.

8)   T   F   For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the identity $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + \vec{u} \times \vec{w}$ holds.

9)   T   F   The set of points which satisfy $-x^2 - 2x + y^2 + z^2 = 1$ define a cone.

10)   T   F   If A, B, C are three points space which are not contained in a common line, then $\vec{AB} \times \vec{AC}$ is a vector orthogonal to the plane containing A, B, C.

11)   T   F   The line $\vec{r}(t) = [t, 5t, 4t]$ hits the plane $x + 5y + 4z = 100$ at a right angle.

12)   T   F   The surface given in (r, θ, z) coordinates as $r = \sin(\theta)$ is a paraboloid.

13)   T   F   If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$, then $\vec{v}$ and $\vec{w}$ are parallel (in the sense that there exists a constant c such that $\vec{v} = c\vec{w}$).

14)   T   F   If $|\vec{x} \times \vec{v}| = 0$ for all vectors $\vec{v}$, then $\vec{x} = \vec{0}$.

15)   T   F   If $\vec{u}$ and $\vec{v}$ are orthogonal, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

16)   T   F   Every vector contained in the line $\vec{r}(t) = [4 + 2t, 2 + 3t, 3 + 4t]$ is parallel to the vector $(4, 2, 3)$.

17)   T   F   If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1/2, 3\pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, -1/2, 0)$.

18)   T   F   The set of points which satisfy $x^2 - 4x + 2y^2 + 3z^2 = -3$ is an ellipsoid.

19)   T   F   If $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$, then all three vectors $\vec{u}, \vec{v}, \vec{w}$ are in the same plane.

20)   T   F   The set of points in $\mathbb{R}^3$ which have distance 1 from the curve $\vec{r}(t) = [3\cos(t), 3\sin(t), 0]$ form a torus (doughnut)
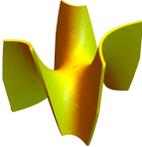
In each sub-problem, each of the numbers 0,1,2,3 each occur exactly once.

a) (2 points) Match the graphs of the functions $f(x, y)$. Enter 0 if there is no match.

1   2   3

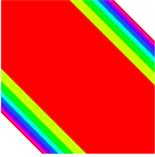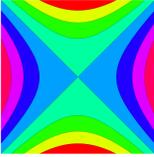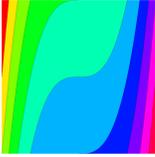| Function $f(x, y) =$	0,1,2, or 3
$xy(x^2 - y^2)$	
$e^{-x^2-y^2}(x^2 + y^2)$	
$1/(x^2 + y^4 + 1)$	
$|\sin(x + y)|$	

b) (2 points) Match the surfaces $g(x, y, z) = c$. Enter 0 if there is no match.

1   2   3

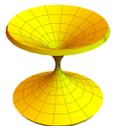| Function $g(x, y, z) =$	0,1,2, or 3
$1000 * \sin(xyz) = 1$	
$x^6 + z^6 = 1$	
$z - (x^2 - y^2)x = 0$	
$x - y^2 = 1$	

c) (2 points) Match the space curves with the parametrizations. Enter 0 if there is no match.

1   2   3

Parametrization $\vec{r}(t) =$	0,1,2, or 3
$[t\cos(t), t\sin(t), \sin(t)]$	
$[\cos(3t), 0, 3\sin(3t)]$	
$[\exp(t), 2\exp(t), 3\exp(t)]$	
$[t, 0, t\sin(t)]$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter 0 if there is no match.

1   2   3

Function $g(x, y) =$	0,1,2, or 3
$x^2 - y^2$	
$(x + y)^4$	
$x^3 - y$	
$\cos(3x) + \sin(3y)$	

e) (2 points) Match the parametrized surfaces. Enter 0 if there is no match.

1   2   3

Parametrization $\vec{r}(u, v) =$	0-3
$[\cos(v), \sin(v), u]$	
$[u^2\cos(v), u^2\sin(v), u]$	
$[\sin(v)\cos(u), \cos(v), 2\sin(v)\sin(u)]$	
$[u^6 + v^6, u^3, v^3]$	

Problem 3) (10 points)

We perform some computations with the vectors $\vec{v} = [2, 2, 3]$ and $\vec{w} = [1, 2, 2]$.

a) (2 points) Find the cross product $\vec{v} \times \vec{w}$?

b) (2 points) Construct a unit vector in the same direction than $\vec{v} \times \vec{w}$.

c) (2 points) Find $\cos(\alpha)$ for the angle α between $\vec{v}, \vec{w}$.

d) (2 points) What is the vector projection $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$?

e) (2 points) Compute $(\vec{v} + \vec{w}) \cdot (\vec{v} \times \vec{w})$.

Problem 4) (10 points)

a) (2 points) Parametrize the plane containing the points $(1, 0, 0), (0, 5, 0), (0, 0, 2)$ using parameters s, t:

b) (3 points) Now find the equation $ax + by + cz = d$ of that plane in a).
c) (5 points) Finally find the distance between that plane defined in a) and the point $P = (3, 3, 3)$.

Problem 5) (10 points)

A **prismatic polyedron** with 8 vertices contains the 4 points $A = (2, 0, 0), B = (1, \sqrt{3}, 0)$ and $C = (-1, \sqrt{3}, 0), D = (0, 0, \sqrt{3})$. Find the distance of the line containing the points A, B and the line containing the points C, D.
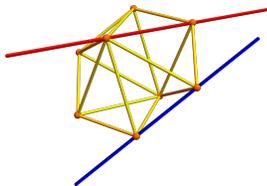
Problem 6) (10 points)

a) (5 points) Find the arc length of the path

$$\vec{r}(t) = [\frac{t^3}{3}, t^2, 2t]$$

with $-1 \le t \le 1$.

b) (5 points) Compute $\vec{v} = \vec{r}\,'(1), \vec{w} = \vec{r}\,''(1)$ and express the curvature of the curve at $\vec{r}(1)$ in terms of $\vec{v}$ and $\vec{w}$.

Problem 7) (10 points)

On Mars, the sum of gravitational acceleration and wind force is $\vec{r}\,''(t) = [0, \sin(t), -4]$. The **Mars helicopter Ingenuity** has been rising up to 5 meters. There had been a small stone pebble stuck on one of the legs. It falls down from the position $\vec{r}(0) = [2, 1, 5]$ with velocity $\vec{r}\,'(0) = [1, 0, 0]$ subject to the acceleration given above.

a) (6 points) Determine the path of the pebble.

b) (4 points) At which time does the pebble hit the ground?

Problem 8) (10 points)

We denote with $|ABC|$ the **area** of a triangle ABC defined by three points A, B, C. Take $A = (2, 0, 0), B = (0, 3, 0), C = (0, 0, 1)$ as well the point $O = (0, 0, 0)$. Verify in this case the **3D Pythagoras theorem**

$$|ABC|^2 = |ABO|^2 + |BCO|^2 + |ACO|^2 .$$

$|ABC|^2 =$

$|ABO|^2 =$

$|BCO|^2 =$

$|ACO|^2 =$

Problem 9) (10 points) No justifications are needed.

4

a) (2 points) Parametrize the surface $3x + 2y + 4z = 12$.

$$\vec{r}(s,t) = [\quad\quad , \quad\quad , \quad\quad]$$

b) (2 points) Parametrize the surface $y^2 + (z-1)^2 = 9$ using an angle θ in the yz-plane.

$$\vec{r}(\theta, x) = [\quad\quad , \quad\quad , \quad\quad]$$

c) (2 points) Parametrize the surface $x^2 + 2x + y^2 + z^2/9 = 0$.

$$\vec{r}(\theta, \phi) = [\quad\quad , \quad\quad , \quad\quad]$$

d) (2 points) Parametrize the surface $y = x^4 - z^4$.

$$\vec{r}(x, z) = [\quad\quad , \quad\quad , \quad\quad]$$

e) (2 points) Parametrize the surface obtained by taking the helix $\vec{r(t)} = [\cos(t), \sin(t), t]$ and connect each point $\vec{r}(t)$ with the projection onto the z-axes.

$$\vec{r}(t, s) = [\quad\quad , \quad\quad , \quad\quad]$$