Checklist III

Triple Integrals

☐ $\iiint_R f(x,y,z)\, dzdydx$ triple integral

☐ $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx$ integral over rectangular box

☐ $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\, dzdydx$ type I region

☐ $\iiint_R f(r,\theta,z)\, \boxed{r}\; dzdrd\theta$ integral in cylindrical coordinates

☐ $\iiint_R f(\rho,\theta,\phi)\, \boxed{\rho^2 \sin(\phi)}\; d\rho d\phi d\theta$ integral in spherical coordinates

☐ $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\, dxdydz$ Fubini

☐ $V = \iiint_E \boxed{1}\, dzdydx$ volume of solid E

Line Integrals

☐ $\vec{F}(x,y) = [P(x,y), Q(x,y)], \vec{F}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]$ vector field.

☐ $\int_C \vec{F} \cdot \vec{dr} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ line integral

☐ $\vec{F}(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals

☐ FTL: $\vec{F}(x,y) = \nabla f(x,y), \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$

☐ closed loop property $\int_C \vec{F}\, \vec{dr} = 0$, for all closed curves C

☐ always equivalent: closed loop property, path independence and gradient field

☐ Clairaut test $\mathrm{curl}(\vec{F}) \neq 0$ assures $\vec{F}$ is not a gradient field

Green's Theorem

☐ $\vec{F}(x,y) = [P,Q]$, curl in two dimensions: $\mathrm{curl}(\vec{F}) = Q_x - P_y$

☐ Green's theorem: C boundary of R, then $\int_C \vec{F} \cdot dr = \int\int_R \mathrm{curl}(\vec{F})\, dxdy$

☐ Take $\vec{F} = [-y, 0]$ or $\vec{F} = [0, x]$ to get area

☐ Green's theorem is useful to compute difficult line integrals or difficult 2D integrals

Flux integrals

☐ $\vec{F}(x,y,z)$ vector field, $S = \vec{r}(R)$ parametrized surface

☐ $\vec{r}_u \times \vec{r}_v$ normal vector, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector

☐ $\vec{r}_u \times \vec{r}_v\, dudv = \vec{dS} = \vec{n}\, dS$ normal surface element

☐ $\int\int_S \vec{F} \cdot \vec{dS} = \int\int_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$ flux integral

Stokes Theorem

$\vec{F}(x,y,z) = [P,Q,R]$, $\mathrm{curl}([P,Q,R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = \nabla \times \vec{F}$

Stokes's theorem: C boundary of surface S, then $\int_C \vec{F} \cdot d\vec{r} = \int \int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$

Stokes theorem allows to compute difficult flux integrals or difficult line integrals

Grad Curl Div

$\nabla = [\partial_x, \partial_y, \partial_z]$, $\vec{F} = \nabla f$, $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$, $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$

$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$ $\qquad$ and $\qquad$ $\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$

$\mathrm{div}(\mathrm{grad}(f)) = \Delta f$ Laplacian

incompressible = divergence free field: $\mathrm{div}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \mathrm{curl}(\vec{G})$

irrotational = $\mathrm{curl}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \mathrm{grad}(f)$

Divergence Theorem

$\mathrm{div}([P,Q,R]) = P_x + Q_y + R_z = \nabla \cdot \vec{F}$

divergence theorem: solid E, boundary S then $\int \int_S \vec{F} \cdot dS = \int \int \int_E \mathrm{div}(\vec{F}) \, dV$

the divergence theorem allows to compute difficult flux integrals or difficult 3D integrals
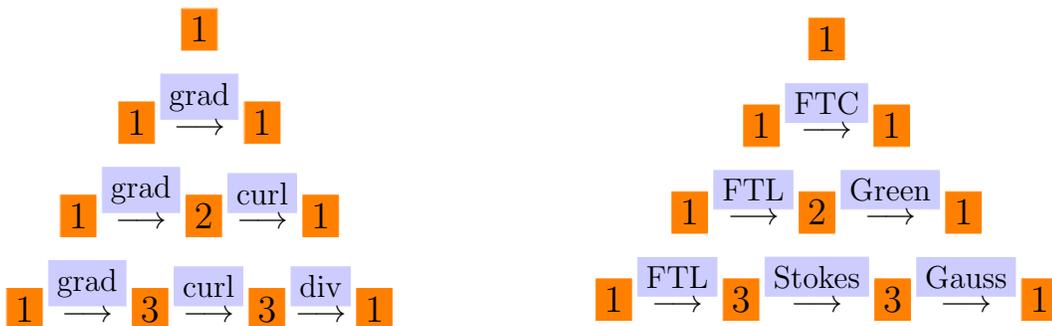
Some topology

interior of a region D: points in D for which small neighborhood is still in D

boundary of a curve: the end points of the curve if they exist

boundary of a surface S points on surface belonging to parameters (u,v) not in the interior of the parameter domain

boundary of a solid D: the surfaces which bound the solid, points in the solid which are not in the interior of D

closed surface: a surface without boundary like for example the sphere

closed curve: a curve with no boundary like for example a knot

Integration overview

Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$

Flux integral: $\int \int_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv$

Theorems and Derivatives overview

1

$1 \xrightarrow{\mathrm{grad}} 1$

$1 \xrightarrow{\mathrm{grad}} 2 \xrightarrow{\mathrm{curl}} 1$

$1 \xrightarrow{\mathrm{grad}} 3 \xrightarrow{\mathrm{curl}} 3 \xrightarrow{\mathrm{div}} 1$

1

$1 \xrightarrow{\mathrm{FTC}} 1$

$1 \xrightarrow{\mathrm{FTL}} 2 \xrightarrow{\mathrm{Green}} 1$

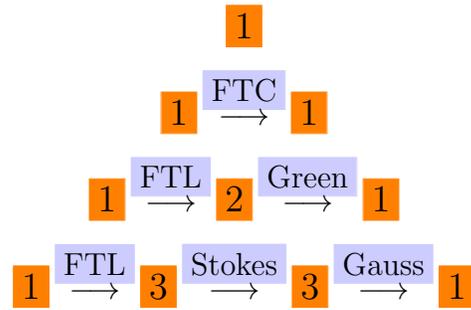$1 \xrightarrow{\mathrm{FTL}} 3 \xrightarrow{\mathrm{Stokes}} 3 \xrightarrow{\mathrm{Gauss}} 1$

Integral theorems overview

All integral theorems have the same form.
One can call it **the fundamental theorem
of multivariable Calculus**:

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the
boundary of G.

$$1$$
$$1 \xrightarrow{\text{FTC}} 1$$
$$1 \xrightarrow{\text{FTL}} 2 \xrightarrow{\text{Green}} 1$$
$$1 \xrightarrow{\text{FTL}} 3 \xrightarrow{\text{Stokes}} 3 \xrightarrow{\text{Gauss}} 1$$

> **Fundamental theorem of line integrals**: if C is an oriented curve with boundary $\{A, B\}$ and f is a function, then
> $$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Remarks.
1) For closed curves, $\int_C \nabla f \cdot d\vec{r} = 0$.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then $\int_A^B \vec{F}\, d\vec{r}$ does not depend on the path connecting A with B.
3) The theorem justifies the name **conservative** for gradient vector fields.
4) The term "potential" was coined by George Green who lived from 1793-1841.

> **1** Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = [\cos(5t), \sin(2t), t^2]$ from $t = 0$ to $t = 2\pi$.
> **Solution**. $\vec{r}(0) = [1, 0, 0]$ and $\vec{r}(2\pi) = [1, 0, 4\pi^2]$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f\, d\vec{r} = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 4\pi^2$.

> **Green's theorem**. If R is a region with oriented boundary C and $\vec{F}$ is a vector field, then
> $$\int \int_R \text{curl}(\vec{F})\, dxdy = \int_C \vec{F} \cdot d\vec{r} \ .$$

Remarks.
1) The curve is oriented in such a way that the region is to the left.
2) The boundary of the curve can consist of piecewise smooth pieces.
3) If $C : t \mapsto \vec{r}(t) = [x(t), y(t)]$, the line integral is $\int_a^b [P(x(t), y(t)), Q(x(t), y(t))] \cdot [x'(t), y'(t)]\, dt$.
4) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
5) If $\text{curl}(\vec{F}) = 0$ everywhere in the plane, then the field is a gradient field.
7) $\vec{F}(x, y) = [0, x]$ we get an **area formula**.
8) Don't mix up Green with Stokes. Green is a theorem in "flatland", where $\vec{F}$ has two components. and $\text{curl}(\vec{F})$ is a scalar.

2 Find the line integral of the vector field $\vec{F}(x,y) = [x^4 + \sin(x) + y, x + y^3]$ along the path $\vec{r}(t) = [\cos(t), 5\sin(t) + \log(1 + \sin(t))]$ with $t \in [0, \pi]$.
Solution. Since $\text{curl}(\vec{F}) = 0$ we can take the simpler path $\vec{r}(t) = [-t, 0]$, $-1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = [-1, 0]$. The line integral is $\int_{-1}^{1} [t^4 - \sin(t), -t] \cdot [-1, 0] \, dt = -t^5/5|_{-1}^{1} = -2/5$.

> **Stokes theorem.** If S is a surface with boundary C and $\vec{F}$ is a vector field, then
>
> $$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \, .$$

Remarks.
1) Stokes theorem implies Green's theorem: if $\vec{F} = [P, Q, 0]$ is z-independent and the surface S is contained in the xy-plane.
2) The orientation of C is such that if you walk along C with head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, of S, then S to your left.
3) Stokes theorem was also found by André Ampère (1775-1836) in 1825. George Stokes (1819-1903) gave it as a multivariable exam problem.
4) The flux of the curl of $\vec{F}$ through S only depends on the boundary C of S.
5) The flux of the curl through a closed surface is zero.

3 Find the line integral of $\vec{F}(x, y, z) = [x^3 + xy, y, z]$ along the polygonal path C connecting $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.
Solution. The path C bounds a surface $S : \vec{r}(u, v) = [u, v, 0]$ parameterized by $R = [0, 2] \times [0, 1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(\vec{F})(x, y, z) = [0, 0, -x]$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = [1, 0, 0] \times [0, 1, 0] = [0, 0, 1]$ so that $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 [0, 0, -u] \cdot [0, 0, 1] \, dudv = \int_0^2 \int_0^1 -u \, dudv = -2$.

> **Divergence theorem**: If S is the oriented boundary of a solid region E in space and $\vec{F}$ is a vector field, then
>
> $$\int\int\int_B \text{div}(\vec{F}) \, dV = \int\int_S \vec{F} \cdot d\vec{S} \, .$$

Remarks.
1) The surface S is oriented so that the normal vector points away from E.
2) The divergence theorem is also called **Gauss theorem**.
3) The theorem allows to determine the flux of vector fields through surfaces.
4) The theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was redis-covered by Carl Friedrich Gauss (1777-1855) and by George Green.
5) For divergence-free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

4 Compute the flux of the vector field $\vec{F}(x, y, z) = [-x, y, z^2]$ through the boundary S of the rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.
Solution. By the divergence theorem, the flux is equal to the triple integral of $\text{div}(\vec{F}) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \, dxdydz = (3 - 0)(2 - (-1))(4 - 1) = 27$.

Integral Theorems Overview

INTEGRATION.

Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

Flux integral: $\iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v\, dudv$

Double integral: $\iint_R f(x,y)\, dxdy.$

Triple integral: $\iiint_E f(x,y,z)\, dxdydz.$

Area $\iint_R 1\, dA = \iint_R 1\, dxdy$

Arc length $\int_a^b |\vec{r}\,'(t)|\, dt$

Surface area $\iint |\vec{r}_u \times \vec{r}_v|\, dudv$

Volume $\iiint_B 1\, dxdydz$

IDENTITIES.

$\text{div}(\text{curl}(\vec{F})) = 0$

$\text{curl}(\text{grad}(f)) = \vec{0}$

$\text{div}(\text{grad}(f)) = \Delta f.$

DIFFERENTIATION.

Velocity: $\vec{r}\,'(t) = d/dt\,\vec{r}(t).$

Partial derivative: $f_x(x,y,z).$

Gradient: $\text{grad}(f) = [f_x, f_y, f_z]$

Curl in 2D: $\text{curl}([P,Q]) = Q_x - P_y$

Curl in 3D: $\text{curl}[P,Q,R] = [R_y - Q_z, P_z - R_x, Q_x - P_y]$

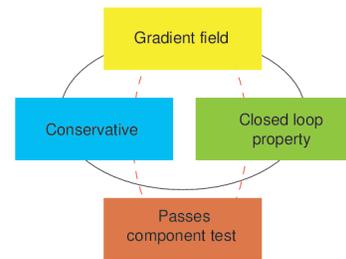**Div:** $\text{div}(\vec{F}) = \text{div}[P,Q,R] = P_x + Q_y + R_z.$

JARGON.

$\text{div}(\vec{F}) = 0$ incompressible

$\text{curl}(\vec{F}) = \vec{0}$ irrotational

CONSERVATIVE FIELDS:

Gradient fields: $\vec{F} = \text{grad}(f).$

Closed curve property: $\int_C \vec{F} \cdot \vec{dr} = 0$ for any closed curve.

Conservative: C_1, C_2 paths from A to B, then $\int_{C_1} \vec{F} \cdot \vec{dr} = \int_{C_2} \vec{F} \cdot \vec{dr}.$

Mixed derivative test: $\text{curl}(F) = 0$ if $\vec{F} = \nabla f.$

If $\vec{F}$ is irotational, then $\vec{F} = \nabla f$ in simply connected R.

TOPOLOGY.

Interior of region D: points which have a neighborhood contained in D.

Boundary of curve: endpoints. **Boundary** of surface: curves. **Boundary** of solid: surfaces.

Simply connected: a closed curve in R can be deformed inside R to a point.

Closed curve Curve without boundary.

Closed surface surface without boundary.

LINE INTEGRAL THEOREM. If $C : \vec{r}(t) = [x(t), y(t), z(t)]$, $t \in [a,b]$ is a curve and $f(x,y,z)$ is a function.

$$\int_C \nabla f \cdot \vec{dr} = f(\vec{r}(b)) - f(\vec{r}(a))$$

GREEN'S THEOREM. If R is a region with boundary C and $\vec{F} = [P,Q]$ is a vector field, then

$$\iint_R \text{curl}(F)\, dxdy = \int_C \vec{F} \cdot \vec{dr}$$

STOKES THEOREM. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot \vec{dr}$$

DIVERGENCE THEOREM. If S is the boundary of a solid E in space with boundary surface S and $\vec{F}$ is a vector field, then

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

GENERAL STOKES. All theorems are of the form

$$\int_G dF = \int_{\delta G} F$$

where dF is the derivative of F and δG is the oriented boundary of G.
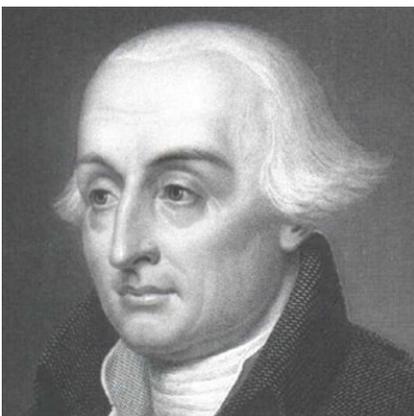
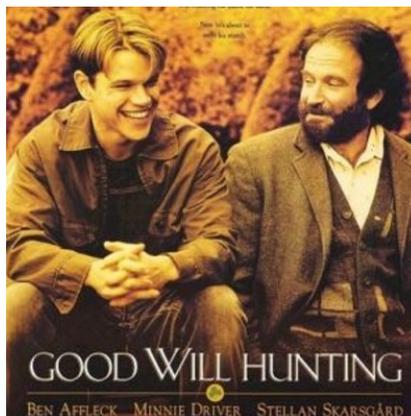George Gabriel Stokes

Mikhail Ostrogradsky

Carl Friedrich Gauss

André-Marie Ampère

Joseph Louis Lagrange

George Green

Last update: summer 2021', Harvard University Math21a, Oliver Knill