

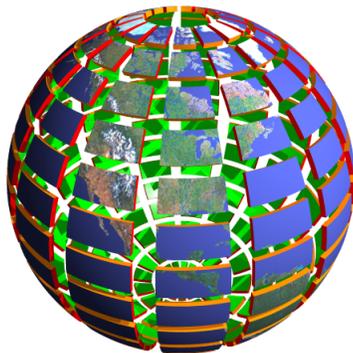
MULTIVARIABLE CALCULUS

MATH S-21A

Unit 18: Spherical integrals

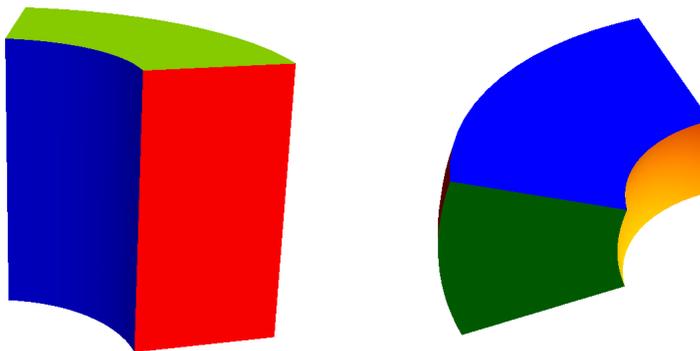
LECTURE

18.1. Cylindrical and spherical coordinate systems help to integrate in many situations.



Definition: Cylindrical coordinates are space coordinates where polar coordinates are used in the xy -plane while the z -coordinate is not changed. The coordinate transformation $T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$, produces the integration factor $[r]$. It is the same factor than what we are used to in polar coordinates.

$$\iiint_{T(R)} f(x, y, z) \, dx \, dy \, dz = \iiint_R g(r, \theta, z) [r] \, dr \, d\theta \, dz$$



Definition: Spherical coordinates use ρ , the distance to the origin as well as two **Euler angles**: $0 \leq \theta < 2\pi$ the polar angle and $0 \leq \phi \leq \pi$, the angle between the vector and the positive z axis. The coordinate change is

$$T : (x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

The integration factor measures the volume of a **spherical wedge** which is $d\rho \cdot \rho \sin(\phi) \cdot d\theta \cdot \rho d\phi = \rho^2 \sin(\phi) d\theta d\phi d\rho$.

$$\iiint_{T(R)} f(x, y, z) \, dx dy dz = \iiint_R g(\rho, \theta, z) \boxed{\rho^2 \sin(\phi)} \, d\rho d\theta d\phi$$

A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin(\phi) \, d\phi d\theta d\rho .$$

The most inner integral $\int_0^\pi \rho^2 \sin(\phi) d\phi = -\rho^2 \cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2 \, d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2 \, d\rho = 4\pi R^3/3$.

Definition: The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int \int \int_G r(x, y, z)^2 \, dz dy dx$, where $r(x, y, z) = \rho \sin(\phi)$ is the distance from the axis L .

EXAMPLES

18.2. For a sphere of radius R we obtain with respect to the z -axis:

$$\begin{aligned} I &= \int_0^R \int_0^{2\pi} \int_0^\pi \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\phi d\theta d\rho \\ &= \left(\int_0^\pi \sin^3(\phi) \, d\phi \right) \left(\int_0^R \rho^4 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \left(\int_0^\pi \sin(\phi)(1 - \cos^2(\phi)) \, d\phi \right) \left(\int_0^R \rho^4 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \\ &= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} . \end{aligned}$$

18.3. If the sphere rotates with angular velocity ω , then $I\omega^2/2$ is the **kinetic energy** of that sphere. The moment of inertia of the earth for example is $8 \cdot 10^{37} \text{kgm}^2$. The angular velocity is $\omega = 2\pi/\text{day} = 2\pi/(86400\text{s})$. The rotational energy is $8 \cdot 10^{37} \text{kgm}^2 / (7464960000\text{s}^2) \sim 10^{29} \text{J} \sim 2.510^{24} \text{kcal}$.

18.4. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.

Solution: we use spherical coordinates to find the center of mass

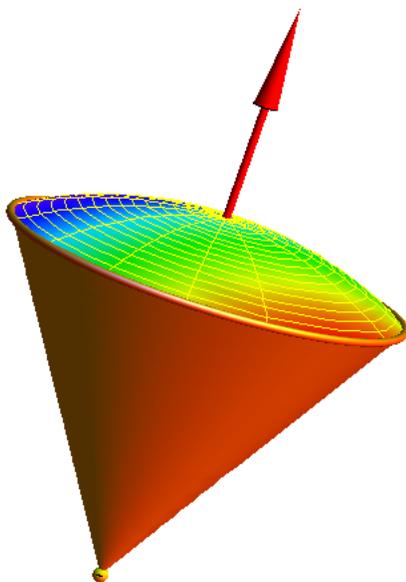
$$\begin{aligned}\bar{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta) d\phi d\theta d\rho \frac{1}{V} = 0 \\ \bar{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta) d\phi d\theta d\rho \frac{1}{V} = 0 \\ \bar{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi) d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}\end{aligned}$$

18.5. Find $\int \int \int_R z^2 dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho \cos(\phi)$, we have

$$\begin{aligned}& \int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi) d\phi d\theta d\rho \\ &= \left(\frac{2^5}{5} - \frac{1^5}{5}\right) 2\pi \left(\frac{-\cos^3(\phi)}{3}\right) \Big|_0^{\pi/4} = 2\pi \frac{31}{5} (1 - 2^{-3/2}).\end{aligned}$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.



Homework

This homework is due on Tuesday, 7/27/2021.

Problem 18.1: Find the moment of inertia $\int \int \int_E (x^2 + y^2) dV$ of the body E whose volume is given by the integral

$$\text{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) d\rho d\theta d\phi .$$

Problem 18.2: Assume the mass density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$ is given by the 8's power of the distance to the z -axis: $\sigma(x, y, z) = r^8 = (x^2 + y^2)^4$. Find its mass

$$M = \int \int \int_E (x^2 + y^2)^4 dx dy dz .$$

Problem 18.3: A solid is described in spherical coordinates by the inequality $\rho \leq 2 \sin(\phi)$. Find its volume.

Problem 18.4: Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

Problem 18.5: Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 4$.

