MULTIVARIABLE CALCULUS

MATH S-21A

Unit 17: Triple integrals

Lecture

17.1. Integrating over 3-dimensional regions is done in the same way than in two dimensions. Three dimensional regions are referred to as **solids**.

Definition: If $f(x, y, z)$ is continuous and E is a **bounded solid** in $\mathbb{R}^3$, then $\iiint_E f(x, y, z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\sum_{(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \in E} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \frac{1}{n^3} \ .$$

Triple integrals can be evaluated by iterated single integrals. Here is an example:

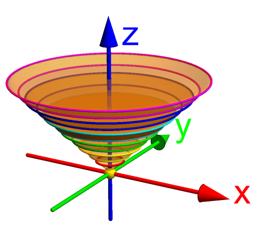
17.2. If E is the box $\{x \in [0, 1], y \in [0, 1], z \in [0, 1]\}$ and $f(x, y, z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

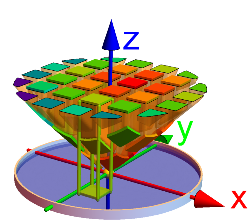
To evaluate the integral, start from the inside $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the most outer layer: $\int_0^1 3x^2 dx$ =1.

For the inner integral, $x = x_0$ and $y = y_0$ are fixed. The middle integral now computes the contribution over a slice $z = z_0$ intersected with R. The outer integral sums up all these slice contributions.

17.3. There are two reductions possible to compute triple integrals:



The **burger method** slices the solid a line and computes $\int_a^b \iint_{R(z)} f(x, y, z)\ dA\ dz$, where $g(z)$ is a double integral giving the values when integrating over cheese, meet or tomato. The **fries method** eats up fries going from $g(x, y)$ to $h(x, y)$ over a region R. We have $\iint_R [\int_{g(x,y)}^{h(x,y)} f(x, y, z)\ dz]\ dA$.



17.4. A special case is the **signed volume**

$$\int\int_R \int_0^{f(x,y)} 1 \; dzdxdy \; .$$

below the graph of a function $f(x, y)$ and above a region R, considered part of the xy-plane. It is the integral $\int\int_R f(x, y) \; dA$. The triple integral above also has more flexibility: we can replace 1 with a function $f(x, y, z)$. If interpreted as a **mass density**, then the integral is the **mass** of the solid.

17.5. The problem of computing volumes has been been worked on by **Archimedes (287-212 BC)** already. His method of exhaustion was a precursor of Riemann sums allowed him to find areas, volumes and surface areas in many cases without calculus. One idea is **comparison**. Already the **Archimedes principle** relating volume to the amount of displayed water is such an idea. The **displacement method** is a **comparison technique**: the area of a sphere is the area of the cylinder enclosing it. The volume of a sphere is the volume of the complement of a cone in that cylinder. **Cavalieri (1598-1647)** would build on Archimedes ideas and determine area and volume using tricks now called the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$. **Newton (1643-1727) and Leibniz(1646-1716)** developed calculus independently. It provided a new tool which made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools. We can do this also in higher dimensions.

Examples

17.6. Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions obtained by solving for z. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

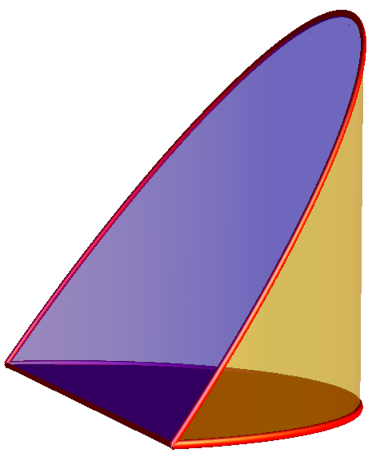
$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1dz] dA \; .$$

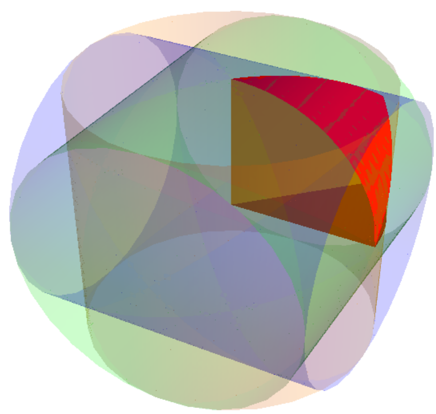
which gives a double integral $\int\int_R 2\sqrt{1 - x^2 - y^2} \; dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi} \int_0^1 \sqrt{1 - r^2} r \; drd\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1 - z^2}$ with area $\pi(1 - z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1 - z^2) \; dz = 4\pi/3$.

17.7. The mass of a body with mass density $\rho(x, y, z)$ is defined as $\int\int\int_R \rho(x, y, z) \; dV$. For bodies with constant density ρ, the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is z. **Solution:**

$$\int_0^2 \int_0^6 \int_0^{4-x^2} z\, dz\, dy\, dx = \int_0^2 \int_0^6 (4-x^2)^2/2\, dydx$$
$$= 6\int_0^2 (4-x^2)^2/2\, dx = 6(\frac{x^5}{5} - \frac{8x^3}{3} + 16x)|_0^2 = 2 \cdot 512/5$$

17.8. The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed a Riemann sum integration technique. It appears in every calculus text book. Find the volume of the hoof. **Solution**. Look from the situation from above and picture it in the xy-plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\, dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta)\, drd\theta = 2/3$.





17.9. Finding the volume of the solid region bound by the three cylinders $x^2+y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is one of the most famous volume integration problems going back to Archimedes.
Solution: look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2\cos^2(\theta)} r\, drd\theta$$

has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x) + \sec(x) + \tan(x)$. The result is $16 - 8\sqrt{2}$.

Homework

This homework is due on Tuesday, 7/27/2021.

Problem 17.1: Evaluate the triple integral
$$\int_0^4 \int_0^z \int_0^{4y} ze^{-2y^2} \; dxdydz \; .$$

Problem 17.2: Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - (x^2 + y^2)$ and satisfying $x \geq 0, y \geq 0$.

Problem 17.3: Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \; dV$ of a cone
$$E = \{x^2 + y^2 \leq z^2 \; 0 \leq z \leq 15 \} \; ,$$
which has the z-axis as its center of symmetry.

Problem 17.4: Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices
$(0, 0, 0), (4, 4, 0), (0, 4, 0), (0, 0, 12)$.

Problem 17.5: This is a classic problem of Archimedes: what is the volume of the body obtained by intersecting the solid cylinders $x^2+z^2 \leq 9$ and $y^2 + z^2 \leq 9$?