MULTIVARIABLE CALCULUS

MATH S-21A

Unit 16: Surface Integration

Lecture

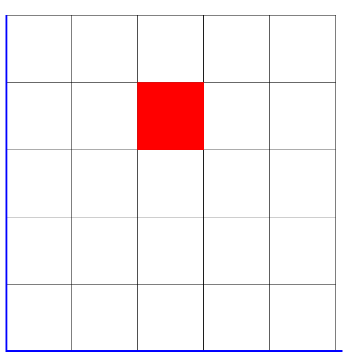
16.1. For certain regions, it is better to use a different coordinate system. A re-parametrization $(x, y) = \vec{r}(u, v)$ often helps. This works then also in higher dimensions, where surfaces are parametrized as $[x, y, z] = \vec{r}(u, v)$. We first remain in $\mathbb{R}^2$, where polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta)$ are an important example.

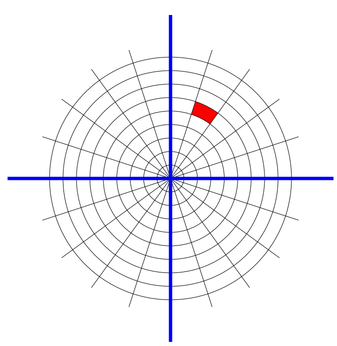
Definition: A **polar region** is a planar region bound by a simple closed curve given in polar coordinates as the curve $(r(t), t)$. In Cartesian coordinates the parametrization of the boundary of a polar region is $\vec{r}(t) = [r(t)\cos(t), r(t)\sin(t)]$, a **polar graph** like the spiral with $r(t) = t$.

Theorem: To integrate in polar coordinates, we evaluate the integral

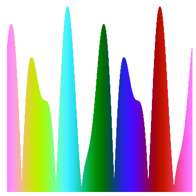
$$\iint_R f(x, y)\, dxdy = \iint_R f(r\cos(\theta), r\sin(\theta))r\, drd\theta\ .$$

16.2. Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with area $dA = d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x, y) plane. It has the area $r\ d\theta dr$. We will also see that the parametrization $\vec{r}(\theta, r) = [r\cos(\theta), r\sin(\theta), 0]$ gives $|\vec{r}_\theta \times \vec{r}_r| = r$.

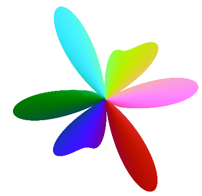




16.3. We can now integrate over basic regions in the (θ, r) plane. Examples are **flowers**: $\{(\theta, r)\ |0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



A polar region shown in polar coordinates.

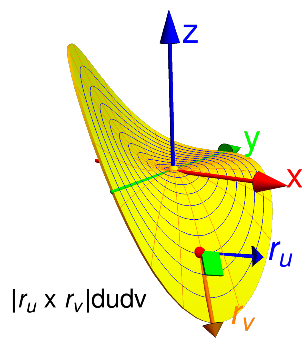


The same region in the xy coordinate system.

Theorem: A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $\vec{r}_u du$ and $\vec{r}_v dv$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv$.

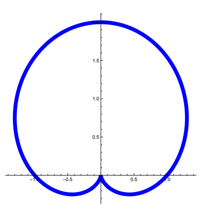


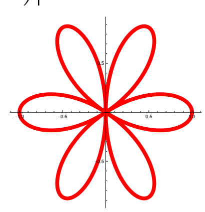
Examples

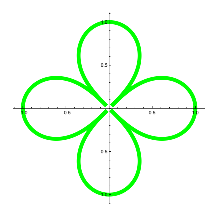
16.4. The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

16.5. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It belongs to the class of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

16.6. The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**.







16.7. Integrate

$$f(x,y) = x^2 + y^2 + xy \; ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\int\int_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

16.8. We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\ \}$ using substitution. It is more elegant to do this integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r\ drd\theta = 2\pi r^2/2|_0^1 = \pi\ .$$

16.9. Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2\ .$$

16.10. Integrate $f(x,y) = y\sqrt{x^2 + y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

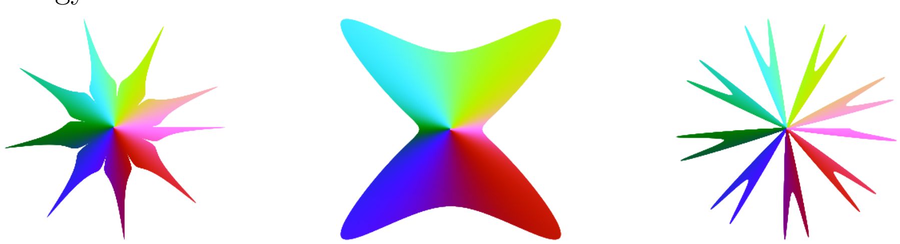
$$\int_1^2 \int_0^{\pi} r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\ d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

16.11. The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This so called **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



16.12. The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

16.13. The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]$ maps the rectangle $G = \{0 \leq x \leq 2\pi, 0 \leq y \leq \pi$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L^2\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1\ dS = \int_0^{2\pi} \int_0^{\pi} L^2\sin(v)\ dvdu = 4\pi L^2$

[1] "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

16.14. For graphs $(u,v) \mapsto [u,v,f(u,v)]$, we have $\vec{r}_u = (1,0,f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u,-f_v,1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1+f_u^2+f_v^2}\, dudv$.

16.15. Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]$ on $R = [0,2\pi]\times[a,b]$. We have $\vec{r}_u = (0,-f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v),\cos(u),\sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v|\, dudv = 2\pi \int_a^b |f(v)|\sqrt{1+f'(v)^2}\, dv$.

Homework

This homework is due on Tuesday, 7/20/2021.

Problem 16.1: Find $\int\int_R (x^2+y^2)^{50}\, dA$, where R is the part of the unit disc $\{x^2+y^2 \leq 1\}$ for which $y > x$.

Problem 16.2: The Cardioid has first been described in 1741. It had become famous because the **main body of the Mandelbrot** set is a Cardioid. It is parametrized by

$$\vec{r}(t) = [\frac{\cos(t)}{2} - \frac{\cos(2t)}{4}, \frac{\sin(t)}{2} - \frac{\sin(2t)}{4}] .$$

Find the area of this region!

Problem 16.3: What is the area of the region which is bounded by the following three curves: first by the polar curve $r(\theta) = \theta$ with $\theta \in [0,2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0,2\pi]$ and third by the positive x-axis?

Problem 16.4: The average of a function f on a region is defined as

$$\frac{\int_R f\, dxdy}{\int_R 1\, dxdy} .$$

Find the average value of $f(x,y) = 2(x^2+y^2)$ on the annular region $R : 1 \leq |(x,y)| \leq 2$.

Problem 16.5: Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 16$.

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