MULTIVARIABLE CALCULUS

MATH S-21A

Unit 13: Extrema

Lecture

13.1. In many applications we are led to the task to **maximize** or **minimize** a function f. As in single variable calculus, we first search for points where the **derivative** is zero because this is needed at maxima by the **Fermat principle**. In one dimensions, like for $f(x) = 3x^5 - 5x^3$ we can use the second derivative test to classify extrema, like the local max at -1 and the local min at 1.

> **Definition:** A point (a, b) in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = [0, 0]$.

13.2. The **Fermat principle** tells here:

> If $\nabla f(x, y)$ is not zero, then (x, y) is not a critical point.

13.3. Proof. Take the directional derivative in the direction $\vec{v} = \nabla f / |\nabla f|$. Then $D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f| > 0$. QED.

13.4. Note that in the definition, we do **not** include points, where f or its derivative is not defined. Without stating otherwise, we always assume that a function f can be differentiated arbitrarily often. Points, where the function has no derivatives are not considered to be part of the domain and need to be studied separately. For the continuous function $f(x, y) = 1/\log(|xy|)$ for example, we would have to look at the points on the coordinate axes as well as the points $xy = 1$ separately.

13.5. In one dimension, we used the condition $f'(x) = 0, f''(x) > 0$ to get a local minimum and $f'(x) = 0, f''(x) < 0$ to assure a local maximum. If $f'(x) = 0, f''(x) = 0$, the nature of the critical point is undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat **inflection point** like for $f(x) = x^3$.

> **Definition:** If $f(x, y)$ is a function of two variables with a critical point (a, b), the number $D = f_{xx} f_{yy} - f_{xy}^2$ is called the **discriminant** of the critical point.

13.6. The discriminant could be remembered better if is seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. As of default, we always assume that functions are twice continuously differentiable. Here is the **second derivative test**:

Theorem: Assume (a,b) is a critical point for $f(x,y)$.
If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum.
If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum.
If $D < 0$ then (a,b) is a saddle point.

13.7. If $D \neq 0$ at all critical points, the function f is called **Morse**. The Morse condition is nice as for $D = 0$, we need higher derivatives or ad-hoc methods to determine the nature of the critical point.

13.8. To determine the maximum or minimum of $f(x,y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see in the next unit how to get extrema on the boundary.

13.9. Sometimes, we want to find the overall maximum and not only the local ones.

Definition: A point (a,b) in the plane is called a **global maximum** of $f(x,y)$ if $f(x,y) \leq f(a,b)$ for all (x,y). For example, the point $(0,0)$ is a global maximum of the function $f(x,y) = 1 - x^2 - y^2$. Similarly, we call (a,b) a **global minimum**, if $f(x,y) \geq f(a,b)$ for all (x,y).

Examples

13.10. Find the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x,y) = [4(x^3 - y), 4(y^3 - x)]$ with critical points $(0,0), (1,1), (-1,-1)$.

13.11. $f(x,y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x,y) = [2x\cos(x^2+y), \cos(x^2 + y) + 1]$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\dots (0,-\pi), (0,\pi), (0,3\pi), \dots$. There are infinitely many.

13.12. The graph of $f(x,y) = (x^2 + y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = [2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2)]e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2 + y^2 = 1$. This function has a continuum of critical points.

13.13. The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = [(0,0]$ for

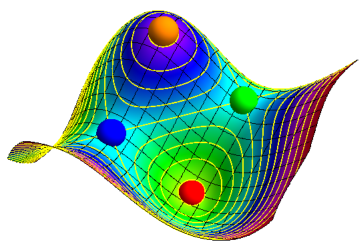
$$(x,y) = \dots, (-\pi,0), (0,0), (\pi,0), (2\pi,0), \dots .$$

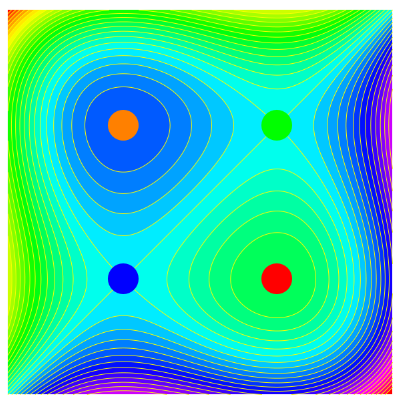
These points are equilibrium points, the angles for which the pendulum is at rest.

13.14. The function $f(x,y) = a\log(y) - by + c\log(x) - dx$ is a function which is invariant by the flow of the **Volterra-Lodka** differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point of f and an equilibrium point of the system.

13.15. The function $f(x, y) = |x| + |y|$ is smooth on the first quadrant $\{x > 0, y > 0\}$. It does not have critical points there. The function has a minimum at $(0, 0)$ but it is not in the domain, where f and ∇f are defined. We have to look at the points on the coordinate axis separately. For $y = 0$, we see that $x = 0$ is a minimum. For $x = 0$ we see that $y = 0$ is a minimum. Indeed $(0, 0)$ is a minimum of f. This minimum was not detected using derivatives.

13.16. The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x, y) = [x^2 - 1, -y^2 + 1]$. There are 4 critical points $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1, 1)$ we have $D = -4$ and so a saddle point,
For $(-1, 1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1, -1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1, -1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





13.17. Find the maximum of $f(x, y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x, -1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and 4/3 is a local maximum on the line $y = -1$. Comparing $f(4/3, 0), f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

13.18. Find the global maxima and minima of $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x, 0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

Homework

This homework is due on Tuesday, 7/20/2021.

Problem 13.1: Find all the extrema of the function

$$f(x,y) = xy + x^2y + xy^2$$

they are maxima, minima or saddle points.

Problem 13.2: Where on the parametrized surface $\vec{r}(u,v) = [1 + u^3, v^2, uv]$ is the temperature $T(x,y,z) = 24y - 24z + 2x + 10$ minimal? To find the minimum, minimize the function $f(u,v) = T(\vec{r}(u,v))$. Find all local maxima, local minima or saddle points of f.

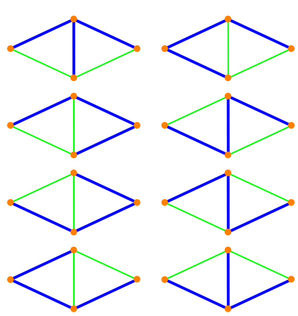
Problem 13.3: Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

Problem 13.4: Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 13e$ and characterize them. Do you find a global maximum or global minimum among them?

Problem 13.5: Graph theorists are fond of at the **Tutte polynomial** $f(x,y)$ of a network. We work with the Tutte polynomial

$$f(x,y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**. Classify using the second derivative test.



Remark. The polynomial is useful: $xf(1-x,0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1,1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1,1) = 8$ as you see the 8 possible trees.

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