**MULTIVARIABLE CALCULUS**

MATH S-21A

# Unit 11: Chain rule

## Lecture

**11.1.** If $f$ and $g$ are functions of a single variable $t$, the **single variable chain rule** tells us that $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$. The rule can be proven by linearizing the functions $f$ and $g$ and verifying the chain rule in the linear case. The **chain rule** is also useful:

**11.2.** To find $\arccos'(x)$ for example, we differentiate $x = \cos(\arccos(x))$ to get $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x))\arccos'(x) = -\sqrt{1-\cos^2(\arccos(x))}\arccos'(x) = -\sqrt{1-x^2}\arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

**Definition:** Define the **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]$.

**11.3.** If $\vec{r}(t)$ is curve and $f$ is a function of several variables we get a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, if $\vec{r}(t)$ is a parametrization of a planar curve $f$ is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

**Theorem:** $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$.

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t),y(t)) = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t) \ .$$

The identity

$$\frac{f(x(t+h),y(t+h))-f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h))-f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h))-f(x(t),y(t))}{h}$$

holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t),y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate $f$ with respect to $t$ and $y$ is treated as a constant:

$$\frac{f(\text{ x(t+h) }) - f(x(t))}{h} = \frac{[f(\text{ x(t) + (x(t+h)-x(t)) }) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} \ .$$

Write $H(t) =$ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t)+H) - f(x(t))]}{H} \cdot \frac{x(t+h)-x(t)}{h} .$$

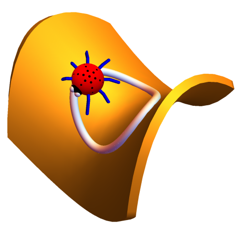
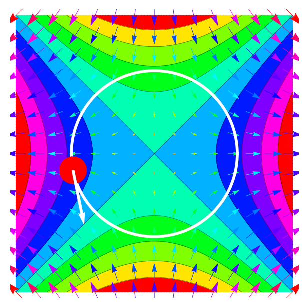
As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.

**11.4.** The chain rule is powerful because it implies other differentiation rules like the addition, product and quotient rule in one dimensions: $f(x,y) = x+y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

**11.5.** As in one dimensions, the chain rule follows from linearization. If $f$ is a linear function $f(x,y) = ax + by - c$ and if the curve $\vec{r}(t) = [x_0 + tu, y_0 + tv]$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

## Examples

**11.6.** A ladybug moves on a circle $\vec{r}(t) = [\cos(t), \sin(t)]$ on a table with temperature distribution $f(x,y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x,y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

**11.7.** From $f(x,y) = 0$, one can express $y$ as a function of $x$, at least near a point where $f_y$ is not zero. From $\frac{d}{dx} f(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain $y' = -f_x/f_y$. Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x,y,z) = c$ defines a surface. Near a point where $f_z$ is not zero, the surface can be described as a graph $z = z(x,y)$. We can compute the derivative $z_x$ without actually knowing the function $z(x,y)$. To do so, we consider $y$ a fixed parameter and compute, using the chain rule

$$f_x(x,y,z(x,y)) \cdot 1 + f_y(x,y,z(x,y)) \cdot 0 + f_z(x,y) \cdot z_x(x,y) = 0$$

so that $z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$. This works at points where $f_z$ is not zero.

**11.8.** The surface $f(x,y,z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$.
**Solution:** $z_x(x,y) = -f_x(2,1,1)/f_z(2,1,1) = -4/(2/9) = -18$.

# Homework

This homework is due on Tuesday, 7/13/2021.

**Problem 11.1:** You know that $d/dt f(\vec{r}(t)) = 35$ at $t = 7$ if $\vec{r}(t) = [t, t]$ and $d/dt f(\vec{r}(t)) = 21$ at $t = 7$. $\vec{r}(t) = [t, 14 - t]$. Find the gradient of $f$ at $(7, 7)$.

**Problem 11.2:** The pressure in the space at the position $(x, y, z)$ is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = [t, t, 1/t]$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

**Problem 11.3:** The chain rule is closely related to linearization. Lets get back to linearization a bit: A farm costs $f(x, y)$, where $x$ is the number of cows and $y$ is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all $x, y$. Estimate $f(12, 19)$.

Here is a song out of this:
*"Old MacDonald had a million dollar farm, E-I-E-I-O,*
*and on that farm he had $x = 10$ cows, E-I-E-I-O,*
*and on that farm he had $y = 20$ ducks, E-I-E-I-O,*
*with $f_x = 2x$ here and $f_y = y^2$ there,*
*and here two cows more, and there a duck less,*
*how much does the farm cost now, E-I-E-I-O?"*

**Problem 11.4:** Find, using implicit differentiation the derivative $d/dx\, \text{arctanh}(x)$, where

$$\tanh(x) = \sinh(x)/\cosh(x) \, .$$

The **hyperbolic sine** and **hyperbolic cosine** are defined as are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. We have $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

**Problem 11.5:** The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines $z$ as a function $z = g(x, y)$ of $x$ and $y$. Find formulas (in terms of $x$,$y$ and $z$) for $g_x(x, y)$ and $g_y(x, y)$. Estimate $g(1.01, 0.99)$ using linear approximation.