

# MULTIVARIABLE CALCULUS

MATH S-21A

## Unit 4: Lines and Planes

### LECTURE

**3.1.** A point  $P = (p, q, r)$  and a vector  $\vec{v} = [a, b, c]$  define the **line**

$$L = \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, t \in \mathbb{R} \right\}.$$

The line consists of all points obtained by adding a multiple of the vector  $\vec{v} = [a, b, c]$  to the vector  $\vec{OP} = [p, q, r]$ . It contains the point  $P$  as well as a copy of  $\vec{v} = \vec{PQ}$  attached to  $P$ . Every vector contained in the line is necessarily parallel to  $\vec{v}$ . We think about the parameter  $t$  as "time". At  $t = 0$ , we are at the end point  $P$  of  $\vec{OP}$  and at  $t = 1$ , we are at the end point  $Q$  of  $\vec{OQ} = \vec{OP} + \vec{v}$ .

**3.2.** If  $t$  is restricted to values in a **parameter interval**  $[t_1, t_2]$ , then  $L = \{[p, q, r] + t[a, b, c], t_1 \leq t \leq t_2\}$  is a **line segment** which connects  $\vec{r}(t_1)$  with  $\vec{r}(t_2)$ . For example, to get the line through  $P = (1, 1, 2)$  and  $Q = (2, 4, 6)$ , form the vector  $\vec{v} = \vec{PQ} = [1, 3, 4]$  and get  $L = \{[x, y, z] = [1, 1, 2] + t[1, 3, 4];\}$ . This can be written also as  $\vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t]$ . If we write  $[x, y, z] = [1, 1, 2] + t[1, 3, 4]$  as a collection of equations  $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$  and solve the first equation for  $t$ :

$$L = \{(x, y, z) \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4\}.$$

**3.3.** The line  $\vec{r} = \vec{OP} + t\vec{v}$  defined by  $P = (p, q, r)$  and vector  $\vec{v} = [a, b, c]$  with nonzero  $a, b, c$  satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c}.$$

The reason is that each of these expressions is equal to  $t$ . These symmetric equations have to be modified a bit one or two of the numbers  $a, b, c$  are zero. If  $a = 0$ , replace the first equation with  $x = p$ , if  $b = 0$  replace the second equation with  $y = q$  and if  $c = 0$  replace third equation with  $z = r$ . The interpretation is that the line is written as an intersection of two planes.

**3.4.** A point  $P$  and two vectors  $\vec{v}, \vec{w}$  define a **plane**  $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers}\}$ .

An example is  $\Sigma = \{[x, y, z] = [1, 1, 2] + t[2, 4, 6] + s[1, 0, -1]\}$ . This is called the **parametric description** of a plane.

**3.5.** If a plane contains the two vectors  $\vec{v}$  and  $\vec{w}$ , then the vector  $\vec{n} = \vec{v} \times \vec{w}$  is orthogonal to both  $\vec{v}$  and  $\vec{w}$ . Because also the vector  $\vec{PQ} = \vec{OQ} - \vec{OP}$  is perpendicular to  $\vec{n}$ , we have  $(Q - P) \cdot \vec{n} = 0$ . With  $Q = (x_0, y_0, z_0)$ ,  $P = (x, y, z)$ , and  $\vec{n} = [a, b, c]$ , this means  $ax + by + cz = ax_0 + by_0 + cz_0 = d$ . The plane is therefore described by a single equation  $ax + by + cz = d$ . We have shown:

**Theorem:** The equation for a plane containing  $\vec{v}$  and  $\vec{w}$  and a point  $P$  is  $ax + by + cz = d$ , where  $[a, b, c] = \vec{v} \times \vec{w}$  and where  $d$  is obtained by plugging in  $P$ .

**3.6. Problem:** Find the equation of a plane which contains the three points  $P = (-1, -1, 1)$ ,  $Q = (0, 1, 1)$ ,  $R = (1, 1, 3)$ .

**Answer:** The plane contains the two vectors  $\vec{v} = \vec{PQ} = [1, 2, 0]$  and  $\vec{w} = \vec{PR} = [2, 2, 2]$ . The normal vector  $\vec{n} = \vec{v} \times \vec{w} = [4, -2, -2]$  leads to the equation  $4x - 2y - 2z = d$ . The constant  $d$  is obtained by plugging in the coordinates of one of the points. In our case, it is  $4x - 2y - 2z = -4$ .

**3.7. Problem:** Find the angle between the planes  $x + y = -1$  and  $x + y + z = 2$ . The **angle between the two planes**  $ax + by + cz = d$  and  $ex + fy + gz = h$  is defined as the angle between the two normal vectors  $\vec{n} = [a, b, c]$  and  $\vec{m} = [e, f, g]$ .

**Answer:** find the angle between  $\vec{n} = [1, 1, 0]$  and  $\vec{m} = [1, 1, 1]$ . It is  $\arccos(2/\sqrt{6})$ .

#### EXAMPLES

**3.8.** To practice the concepts, we look at **distance formulas**.

1) If  $P$  is a point and  $\Sigma : \vec{n} \cdot \vec{x} = d$  is a plane containing a point  $Q$ , then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between  $P$  and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from  $P = (7, 1, 4)$  to  $\Sigma : 2x + 4y + 5z = 9$ , we find first a point  $Q = (0, 1, 1)$  on the plane. Then compute

$$d(P, \Sigma) = \frac{|[-7, 0, -3] \cdot [2, 4, 5]|}{|[2, 4, 5]|} = \frac{29}{\sqrt{45}}.$$

2) If  $P$  is a point in space and  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$ , then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between  $P$  and the line  $L$ . Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from  $P = (2, 3, 1)$  to

the line  $\vec{r}(t) = (1, 1, 2) + t(5, 0, 1)$ , compute

$$d(P, L) = \frac{|[-1, -2, 1] \times [5, 0, 1]|}{|[5, 0, 1]|} = \frac{|[-2, 6, 10]|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}.$$

3) If  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$  and  $M$  is the line  $\vec{s}(t) = P + t\vec{v}$ , then

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines  $L$  and  $M$ . Proof: the distance is the length of the vector projection of  $\vec{PQ}$  onto  $\vec{u} \times \vec{v}$  which is normal to both lines. For example, to compute the distance between  $\vec{r}(t) = (2, 1, 4) + t(-1, 1, 0)$  and  $M$  is the line  $\vec{s}(t) = (-1, 0, 2) + t(5, 1, 2)$  form the cross product of  $[-1, 1, 0]$  and  $[5, 1, 2]$  is  $[2, 2, -6]$ . The distance between these two lines is

$$d(L, M) = \frac{|(3, 1, 2) \cdot (2, 2, -6)|}{|[2, 2, -6]|} = \frac{4}{\sqrt{44}}.$$

4) To get the distance between two planes  $\vec{n} \cdot \vec{x} = d$  and  $\vec{n} \cdot \vec{x} = e$ , then their distance is

$$d(\Sigma, \Pi) = \frac{|e - d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example,  $5x + 4y + 3z = 8$  and  $10x + 8y + 6z = 2$  have the distance

$$\frac{|8 - 1|}{|[5, 4, 3]|} = \frac{7}{\sqrt{50}}.$$



FIGURE 1. The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites.

HOMEWORK

This homework is due on Tuesday, 6/29/2021.

**Problem 4.1:** Given the three points  $P = (7, 4, 5)$  and  $Q = (1, 3, 9)$  and  $R = (4, 2, 10)$ , find the parametric equation for the line perpendicular to the triangle  $PQR$  passing through its center of mass  $(P + Q + R)/3 = (4, 3, 8)$ .

**Problem 4.2:** A regular tetrahedron has vertices at the points  $P_1 = (0, 0, 6), P_2 = (0, \sqrt{32}, -2), P_3 = (-\sqrt{24}, -\sqrt{8}, -2)$  and  $P_4 = (\sqrt{24}, -\sqrt{8}, -2)$ . Find the distance between two edges which do not intersect.

**Problem 4.3:** Find a parametric equation for the line through the point  $P = (3, 1, 2)$  that is perpendicular to the line  $L : x = 1 + 4t, y = 1 - 4t, z = 8t$  and intersects this line in a point  $Q$ .

**Problem 4.4:** Given three spheres of radius 9 centered at  $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$ . Find a plane  $ax + by + cz = d$  which touches all of three spheres from the same side.

**Problem 4.5:** a) Find the distance between the point  $P = (3, 3, 4)$  and the line  $2x = 2y = 2z$ .  
 b) Parametrize the line  $\vec{r}(t) = [x(t), y(t), z(t)]$  in a) and find the minimum of the function  $f(t) = d(P, \vec{r}(t))^2$ . Verify that the minimal value agrees with a).

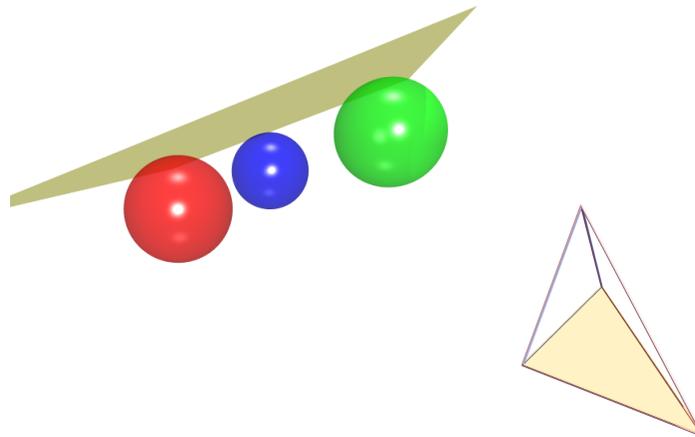


FIGURE 2. The sphere problem and the tetrahedron.