MULTIVARIABLE CALCULUS

MATH S-21A

Unit 2: Vectors and dot product

Lecture

2.1. Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$.

We write this column vector also as a row vector $[x - a, y - b, z - c]$ in order to save space. As the vector starts at P to Q we write $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in $\vec{v} = [p, q, r]$ are called the **components** of $\vec{v}$.

2.2. Vectors can be attached to any point in space. Two vectors with the same components are considered **equal** as they can be translated into each other. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]$ heads to the point (p, q, r). One could therefore identify **points** $P = (a, b, c)$ with **vectors** $\vec{v} = [a, b, c]$ attached to the origin. For clarity, we often draw an arrow $\vec{}$ on top of a vector variable and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to write $[2, 3, 4]$ for vectors and $(2, 3, 4)$ for points.

2.3.

Definition: The **sum** of two vectors is $\vec{u}+\vec{v} = [u_1, u_2]+[v_1, v_2] = [u_1+v_1, u_2+v_2]$. The **scalar multiple** is $\lambda\vec{u} = \lambda[u_1, u_2] = [\lambda u_1, \lambda u_2]$. The **difference** $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

Commutativity, associativity, or distributivity rules for vectors are inherited directly from the corresponding rules for numbers.

2.4. The vectors $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$, $\vec{k} = [0, 0, 1]$ are called **standard basis vectors**. This has historically grown because the dot and cross product have grown from **quaternions** which are points (t, x, y, z) in $\mathbb{R}^4$, usually written as $q = t + ix + jy + kz$.

2.5.

Definition: The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is called a **direction** of $\vec{v}$. The only vector of length 0 is the 0 vector $[0, 0, 0]$.

2.6.

Definition: The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

2.7. Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, the **Dirac notation** $\langle \vec{v} | \vec{w} \rangle$ is used in quantum mechanics or the **Einstein notation** $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity is used. In statistics, it is called the **covariance** $\text{Cov}[v, w]$ of centered data points. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product $g(v, w)$ which is linear in v and w and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v, w) = 2v_1w_1 + 3v_2w_2 + 5v_3w_3$.

2.8. The dot product determines distances and distances determines the dot product.
Proof: Write $v = \vec{v}$. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v + w) \cdot (v + w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

$$v \cdot w = (|v + w|^2 - |v|^2 - |w|^2)/2 \, .$$

2.9. The **Cauchy-Schwarz inequality** is

Theorem: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. If $|w| = 0$, the statement holds as both sides are zero. Otherwise, assume $|w| = 1$ by dividing the equation by $|w|$. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw) \cdot (v - aw)$ to get $0 \leq (v - (v \cdot w)w) \cdot (v - (v \cdot w)w) = |v|^2 + (v \cdot w)^2 - 2(v \cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$. □

2.10. Having established this, it is possible to give a definition of what an **angle** is, without referring to any geometric pictures:

Definition: The **angle** between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Since cos maps $[0, \pi]$ in a 1:1 manner to $[-1, 1]$, this is well defined.

2.11. The **Al Kashi's theorem** gives the third side length c of a triangle ABC in terms of the sides $a = d(B, C), b = d(A, C)$ and α, the angle at the vertex C

Theorem: $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$. □

2.12. The **triangle inequality** tells

Theorem: $|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof. $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}^2+\vec{v}^2+2\vec{u}\cdot\vec{v} \leq \vec{u}^2+\vec{v}^2+2|\vec{u}\cdot\vec{v}| \leq \vec{u}^2+\vec{v}^2+2|\vec{u}|\cdot|\vec{v}| = (|\vec{u}|+|\vec{v}|)^2$. □

Definition: Two vectors are called **orthogonal** or **perpendicular** if $\vec{v}\cdot\vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2,3]$ is orthogonal to $\vec{w} = [-3,2]$.

2.13. We can now prove the **Pythagoras theorem**:

Theorem: If $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v-w|^2 = |v|^2 + |w|^2$.

Proof. $(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}+2\vec{v}\cdot\vec{w} = \vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}$. □

2.14.

Definition: The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.

2.15. The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Examples

2.16. For example, with $\vec{v} = [0,-1,1]$, $\vec{w} = [1,-1,0]$, $\mathrm{P}(\vec{v}) = [1/2,-1/2,0]$. Its length is $1/\sqrt{2}$.

2.17. The RGB **color space** consists of triples $\vec{v} = [r,g,b]$ describing the amount of red, green and blue of a **color**. An other coordinate system is the CMY **color space** consisting of triples $\vec{v} = [c,m,y] = [1-r,1-g,1-b]$, where c is **cyan**, m is **magenta** and y is **yellow**.

2.18. In physics, forces and fields $\vec{F}$ are described by vectors. The **velocity** of a curve $r(t) = [x(t),y(t),z(t)]$ is a vector attached to the point $r(t)$.

2.19. In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear.

Homework

This homework is due on Tuesday, 6/29/2021.

Problem 2.1: a) Find a **unit vector** parallel to $\vec{x} = \vec{u} + \vec{v} + \vec{w}$ if $\vec{u} = [1,0,1]$ and $\vec{v} = [1,1,0]$ and $\vec{w} = [0,1,1]$.
b) Now find a two non-parallel unit vectors perpendicular to $\vec{x}$.

Problem 2.2: An **Euler brick** is a **cuboid** with side lengths a, b, c such that all face diagonals are integers.
a) Verify that $\vec{v} = [a,b,c] = [275, 252, 240]$ is a vector which leads to an Euler brick. Halcke found the first one in 1719.
b) (*) Verify that $[a,b,c] = [u(4v^2 - w^2), v(4u^2 - w^2), 4uvw]$ leads to an Euler brick if $u^2 + v^2 = w^2$.
(Sounderson 1740) If also the space diagonal $\sqrt{a^2+b^2+c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

Problem 2.3: **Colors** are encoded by vectors $\vec{v} =$ [red , green , blue]. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0,1]$.
a) Determine the angle between the colors yellow and magenta.
b) What is the vector projection of the magenta-orange mixture $\vec{x} = (\vec{v} + \vec{w})/2$ onto green $\vec{y}$?

Problem 2.4: A rope is wound exactly 8 times around a stick of circumference 1 and length 15. How long is the rope?

Problem 2.5: a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.
b) Find the vector projection of the main diagonal $\vec{v} = [1,1,1]$ onto the side diagonal $\vec{w} = [1,1,0]$.
c) Find the maximal distance between the 16 points $(\pm1, \pm1, \pm1, \pm1)$ of a **tesseract**.