8/5/2021 PRACTICE EXAM 5 **Maths 21a, O. Knill, Summer 2021**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) No justifications are necessary

1) [✓ T] F The parametrization $\vec{r}(u,v) = [v\cos(u), v, v\sin(u)]$ describes a cone.

Solution:
Indeed $x^2 + z^2 = y^2$

2) T [✓ F] The projection vector $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$ always has length smaller or equal than the length of $\vec{w}$.

Solution:
The projection vector is independent of the length of $\vec{w}$.

3) T [✓ F] The vectors $\vec{v} = [1, 2, 3]$ and $\vec{w} = [3, 2, 1]$ are parallel.

Solution:
Their cross product is not zero.

4) [✓ T] F Let S is the unit sphere and $\vec{F}$ is a vector field in space satisfying $\text{div}(\vec{F}) = 0$ everywhere, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
By the divergence theorem.

5) T [✓ F] If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\int_C \vec{F} \cdot d\vec{r}(t) = 0$ for any closed curve C.

Solution:
The flux integrals though a closed surface would be zero

6) T [✓ F] If $\vec{F}(x,y,z)$ has zero divergence everywhere in space, then $\vec{F}$ has zero curl everywhere in space.

Solution:
Get an example, like $[-y, x, 0]$.

7) T [✓ F] If $\vec{F}, \vec{G}$ are two vector fields which have the same curl then $\vec{F} - \vec{G}$ is a constant vector field.

Solution:
Take $[x, 0, 0], [0, y, 0]$. They both have curl zero.

8) T [✓ F] As you know, we also write $\text{grad}(f) = \nabla f$ for the gradient. In $\mathbf{R}^3$ the equation $\text{grad}(\text{curl}(\text{grad}(f))) = \vec{0}$ always holds.

Solution:
Since the curl of the gradient is is the zero vector $\vec{0}$, this reads like $\text{grad}(\vec{0}) = \vec{0}$ which does not make any sense as the gradient is only defined for scalar functions.

9) T [✓ F] The linearization of $f(x, y) = 1 + 3x$ is the function $L(x, y) = 3x$.

Solution:
The linearization is $1 + 3x$.

10) [✓ T] F If $\vec{r}(t), a \leq t \leq b$ is part of a flow line of a vector field $\vec{F}(x, y, z)$ for which $|\vec{F}(x, y, z)| = 1$ at every point, then $|\int_a^b \vec{F}(\vec{r}(t))\, dt|$ is the arc length.

Solution:
We have $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \vec{r}\,(t)$.

11) T [✓ F] There is a non-constant function $f(x, y, z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$.

Solution:
The right hand side is 0.

12) [✓ T] F If $\vec{F}$ is a vector field and E is the solid region $x^4 + y^4 + z^4 \leq 1$, then $\iiint_E \text{div}(\text{curl}(\vec{F}))\, dxdydz = 0$.

Solution:
Because $\text{div}(\text{curl}(\vec{F}) = \vec{0}$.

13) [✓ T] F

If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is the volume of the enclosed solid.

Solution:
By the divergence theorem.

14) T [✓ F]

The vector $\vec{j} \times (\vec{j} \times \vec{i})$ is the zero vector, if $\vec{i} = [1,0,0], \vec{j} = [0,1,0]$, and $\vec{k} = [0,0,1]$.

Solution:
$\vec{j} \times \vec{i} = -\vec{k}$ which is perpendicular to $\vec{j}$. So, the cross product is parallel to $\vec{i}$ and non-zero.

15) [✓ T] F

If $f(x,y)$ is maximized at (a,b) under the constraint $g = c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel.

Solution:
These are the Lagrange equations

16) T [✓ F]

The distance between a point P and the line L through two different points A, B is given by the formula $|\vec{PA} \cdot \vec{AB}|/|\vec{AB}|$.

Solution:
One has to use a cross product.

17) T [✓ F]

There exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = \vec{F}$.

Solution:
It is not possible, as the divergence is a scalar.

18) [✓ T] F

The unit tangent vector $\vec{T}(t)$ is perpendicular to the vector $\vec{T}\,'(t)$.

Solution:
We have shown that in class.

19) [✓ T] F The vector field $\vec{F}(x, y, z) = [x, 2x, 3x]$ can not be the curl of an other vector field.

Solution:
Its divergence is not zero

20) [✓ T] F The expression div(grad(div(curl(curl(grad(f)))))) is a well defined function of three variables if f is a function of three variables.

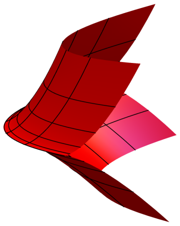
Solution:
Every of the expressions make sense. Not well defined would be grad(grad(f)) for example as the gradient is only defined for scalar functions.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match

Parametrized surface $\vec{r}(u,v)$	1-3
$[u, v, \exp(-(u^2+v^2))]$	
$[\sin(v+u), \cos(u-v), \sin(u)]$	
$[v^2+u^2, u, v^3]$	

1



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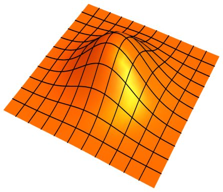

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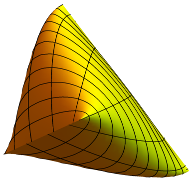
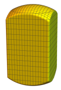
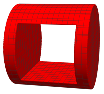


b) (2 points) Match the solids. There is an exact match.

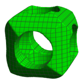
Solid	a-c
$x^2 < 8, y^2+z^2 < 9 \ (x^2 > 4 \text{ or } z^2 > 4)$	
$1 \leq x^2+z^2 \leq 4, 1 \leq x^2+y^2 \leq 4, 1 \leq y^2+z^2 \leq 4$	
$x^2+y^2+z^2 \leq 4, x^2 < 1, y^2 < 1$	



a



b

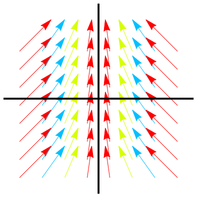


c

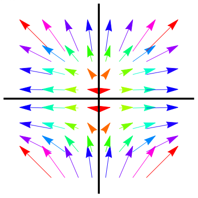
c) (2 points) The figures display vector fields. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = [x, y^2]$	
$\vec{F}(x,y) = [-x, 1]$	
$\vec{F}(x,y) = [x^2, y^2]$	
$\vec{F}(x,y) = [y^2, x]$	

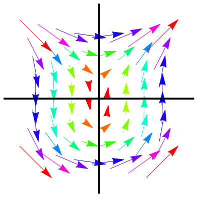
I



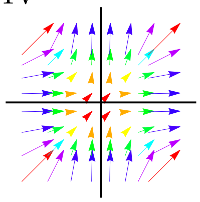
II



III



IV



d) (2 points) Hunt down those partial differential equations!

Equation	A-D or 0 if no match
Clairaut	
Burger	
Wave	
Laplace	

	Partial differential equations
A	$u_{tt} - u_{xx} = 0$
B	$u_{tt} + u_{xx} = 0$
C	$u_t + uu_x = 0$
D	$u_t + u_x = 0$

e) (2 points) In the second midterm, some creative nicknames were created for Fubini: we saw "Fudini", "Fumini", "The Italian guy", "The French guy". Some even called him "Clairaut". Lets see again: you task is to chose from the following names "Euler", "Hamilton", "Frenet", "Clairaut", "Fubini", "Cauchy", "Schwarz", "Laplace", "Stokes", "Gauss", "Green", "Ostogradsky", "Archimedes", "Lagrange", "Fermat".

	Name of person or pair of persons
Who showed $f_{xy} = f_{yx}$?	
Who proved $\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\|\|\vec{w}\|$?	

Another joker! We have mentioned the **Maxwell equations** in the Stokes lecture class. Here are three of these equations. The fourth one is still missing. If you can write down the missing one, you can regain 2 points, possibly lost earlier in this problem.

$$\boxed{\text{div}(\vec{E}) = \sigma, \text{curl}(\vec{E}) = -\vec{B}_t, \ \text{curl}(\vec{B}) = j + \vec{E}_t}$$

Solution:
2,3,1
b,c,a
II,I,IV,III
O,C,A,B
Caliraut, Cauchy-Schwarz
div(B)=0 (no monopoles).

Problem 3) (10 points) No justifications necessary

a) (5 points) Complete the formulas:

Formula	where X is
$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\mathbf{X}$	
$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\mathbf{X}$	
$\vec{P}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w})/\mathbf{X}$	
$\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ if and only if $\mathbf{X}$ is zero	
If $\vec{v} \times \vec{w} = 0$, then $\vec{v}$ and $\vec{w}$ are $\mathbf{X}$	

b) (5 points) Choose from the following words to complete the table below:

"arc length formula", "surface area formula", "chain rule", "volume", "area of parallelogram", "partial differential equation", "Fubini Theorem", "line integral", "flux integral", "Phytagoras theorem", "Al Khashi cos-formula", "vector projection", "scalar projection", "partial derivative", "unit tangent vector", "normal vector", "binormal vector"

Formula	Name of formula or rule or theorem
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$	
$\int_a^b \lvert \vec{r}\,'(t) \rvert \; dt$	
$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$	
$\lvert\vec{v} - \vec{w}\rvert^2 = \lvert\vec{w}\rvert^2 + \lvert\vec{v}\rvert^2 - 2\lvert\vec{v}\rvert\lvert\vec{w}\rvert \cos(\theta)$	
$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$	

Solution:
a) $\cos(\theta)$
$\sin(\theta)$
$|w|^2$
the Volume
parallel.
b) Chain rule
Length
PDE
Al Khashi cos formula
Normal vector

Problem 4) (10 points)

A recent viral video features **Otmashka Duminina** sky diving from the **Aizhai bridge** in China. The clip became famous because she first had to get rid of police officers who were trying to prevent her jump, wink them good-by and then jump. There is some wind. The acceleration is

$$\vec{r}\,''(t) = [1+t, t, -10]\ .$$

Source: Youtube

a) (5 points) Write down $\vec{r}(t)$ and find the position $\vec{r}(2)$, if $\vec{r}(0) = [0,0,300]$ and $\vec{r}\,'(0) = [1,0,0]$.

b) (5 points) Find the curvature

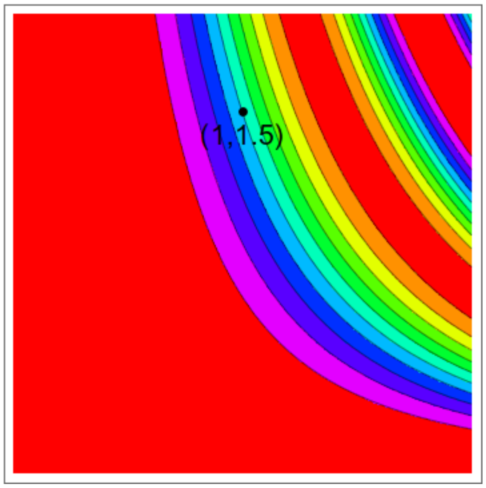
$$\frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

of the path at $\vec{r}(2)$.

Solution:
a) $\vec{r}(t) = [tt^2/2 + t^3/6, t^3/6, -5t^2 + 300]$, $\vec{r}(2) = [16/3, 4/3, 280]$.
b) $r'(\vec{2}) = [5,2,-20]$, $\vec{r}''(2) = [3,2,-10]$. We have $\vec{r}'(2) \times \vec{r}''(2) = [20,-10,4]$. The curvature is $\boxed{516^{1/2}/(429)^{3/2}}$.

Problem 5) (10 points)

In this problem, we work with the function $f(x,y) = \cos(x^2y)$.



a) (4 points) Estimate $f(1.001, \pi/2 - 0.0001)$ using linear approximation.

b) (3 points) Find the tangent plane to the surface $g(x,y,z) = \cos(x^2y) - z^2 + 2z = 0$ at $(1, \pi/2, 2)$.

c) (3 points) Find the direction = unit vector in which the directional derivative $D_{\vec{v}}f(1,\pi/2)$ is maximal.

Solution:
a) $\nabla f(x,y) = [-\sin(x^2y)2xy, \sin(x^2y)x^2]$. At the given point this is $\nabla f(x,y) = [-\pi, -1]$. We estimate $0 - \pi 0.001 + 0.0001$.
b) $\pi x + y + 2z = 3\pi/2 + 4$.
c) $[-\pi, -1]/\sqrt{1+\pi^2}$.

Problem 6) (10 points)

While searching for a geometry problem, Oliver sat at the **Harvard square coffee-shop** just in front of spherical lamps located at

$$A = (1,2,4), B = (2,2,3), C = (3,1,5) .$$

Well, here it is, the geometry problem:

a) (3 points) Find the area of the triangle ABC.

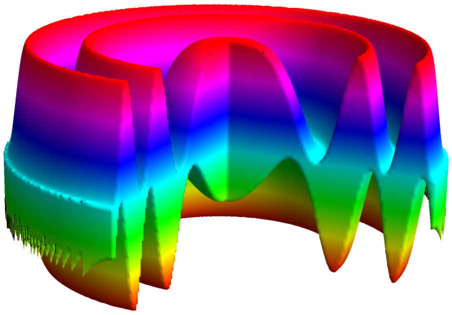
b) (3 points) Find the equation of the plane through A,B and C.

b) (4 points) Find the distance of C to the line through the points A and B.

Foto by Oliver taken on August 2, 2016

Solution:
a) The area of the triangle is half of the area $|\vec{AC} \times \vec{AB}|$ of the parallel epiped spanned by the vectors $\vec{AC}$ and $\vec{AB}$. The area is $\sqrt{11}/2$
b) We have the vectors $\vec{AC} \times \vec{AB} = \vec{n} = [2,-1,1] \times [1,0,-1] = [1,3,1]$. This gives the equation $x + 3y + z = d$. The constant is obtained by plugging in a point. It is $x + 3y + z = 11$.
c) The distance is the area of the parallelogram ($\sqrt{11}$) divided by the base length $\sqrt{2}$. It is $\boxed{\sqrt{11/2}}$.

Problem 7) (10 points)



Find the volume of the solid given by the inequalities

$$x^2 + y^2 \leq 16$$

$$\sin(x^2 + y^2) \leq z \leq 2 + \cos(x^2 + y^2)$$

$$x \geq 0 \quad \text{or} \quad y \geq 0 \ .$$

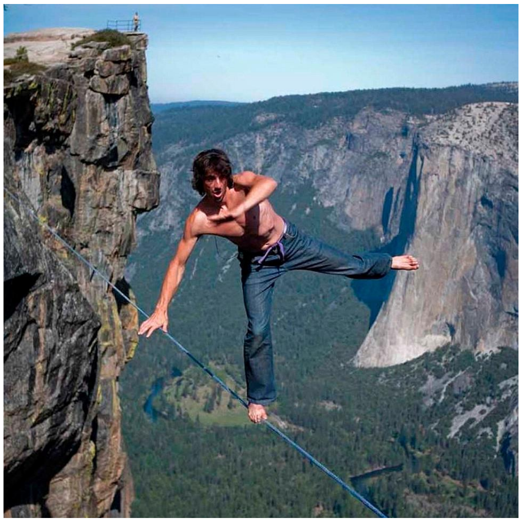
As the picture illustrates, the last condition means that the points (x, y, z) for which $x < 0, y < 0$ are excluded.

Solution:
After evaluating the most inner integral, we get to the following integral in polar coordinates:

$$\int_{\pi}^{5\pi/2} \int_{0}^{4} (2 + \cos(r^2) - \sin(r^2)) r \; dr d\theta \ .$$

This gives $\boxed{3\pi/4(31 + \sin(16) + \cos(16))}$. It was also possible to integrate from 0 to 2π and take $3/4$ of the result.

Problem 8) (10 points)



Dean Potter solo walking at Taft Point in Yosemite

We like it **"extreme"**. Maximize the function

$$f(x, y) = x^{1000} - 1000x + y^{200} - 100y^2 \ .$$

To do so, find all critical points and classify them using the second derivative test. This is an interesting example, as computer algebra systems have trouble writing down the solutions (they look for solutions in the complex). But we can find the real ones!

Solution:
The gradient of f is $[1000(x^{999}-1), 200y^{199}-200y]$. It is zero for $x=1$ and $y=0,1,-1$. There are three solutions. $(1,0)$ is a saddle, $(1,1),(1,-1)$ are minima. In order to use the second derivative test, we need $D = f_{xx}f_{yy} - f_{xy}^2$. We have $f_{xx} = 1000 \cdot 999x^{998}$ which is $1000 \cdot 999$ in all three cases. Now, $f_{yy} = 200 \cdot 199y^{198} - 200$. This is -200 for y and otherwise positive. So, we see that D is negative for $(1,0)$ and otherwise $D > 0$. As $f_{xx} > 0$ in all the remaining cases, these are all minima.
P.S. What is interesting is that when Mathematica computes the critical points, it get $1000 \cdot 200$ critical points in the complex. The output consists of 200'000 entries, this is one reason why even a computer algebra system starts to stutter, especially if we trained it to write down a neat table! But we like it extreme!

Problem 9) (10 points)

a) (5 points) On the other extreme side, here is one of the simplest Lagrange problems: extremize

$$f(x,y) = x^2 - y$$

on the curve

$$g(x,y) = y - x - 1 = 0$$

using the Lagrange method.
b) (5 points) **Economists** also solve Lagrange problems by finding the critical points of $F(x,y,z) = f(x,y) - zg(x,y)$ of three variables. (z plays the role of the Lagrange multiplier.) Write down $F(x,y,z)$ for the example given in a) and find its critical points $\nabla F(x,y,z) = 0$. You should get the same x,y values as in a).

Solution:
a) The Lagrange equations are $2x = -\lambda, -1 = \lambda 1$ which gives immediately $x = 1/2$. Plugging this into the constraint gives $y = 3/2$. So, the critical point under constraint is $\boxed{(1/2, 3/2)}$.
b) The gradient of $F(x,y,z)$ is $\nabla F(x,y,z) = [f_x - zg_x, f_y - zg_y, -g(x,y)]$. If we set this to $[0,0,0]$, we which are exactly the Lagrange equations from a) where λ is replaced with z. Again, we get $\boxed{(1/2, 3/2, -1)}$.

Problem 10) (10 points)

Three days ago, on August 1, 2021, Lukas Irmler walked over a rope over the **Rheinfalls** in Switzerland. There is a force field $\vec{F}$ present which consists part of the gravitational force and part by the wind forces:

$$\vec{F}(x, y, z) = [\sin(x), \cos(y), -10 + z] \ .$$

The path is given by $\vec{r}(t) = [5t, t, 30 - \sin(t)/10]$, where $0 \leq t \leq \pi$. Compute the work

$$\int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \ dt$$

done by Lukas during this stunt.

Source: Schaffhauser Nachrichten, August 2, 2021

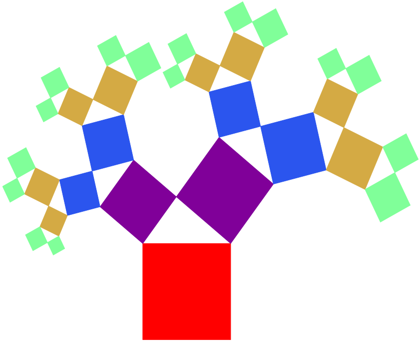
Solution:
Best use the FTLI. The potential is $f(x, y, z) = -\cos(x) + \sin(y) - 10z + z^2/2$. Instead of integrating, we just have to evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(5\pi, \pi, 30) - f(0, 0, 30) = 2$. It was also possible to do the line integral directly, but it was considerably more work.

Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x, y) = [-y + x^8, x - y^9]$$

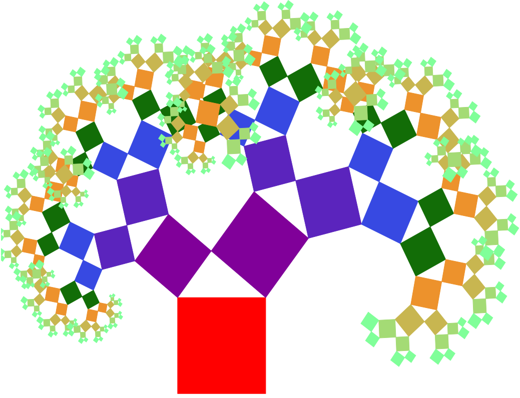
along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



Pythagoras tree theorem:

$\boxed{\text{The generation } n \text{ Pythagorean tree has area } n+1.}$

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.



Solution:

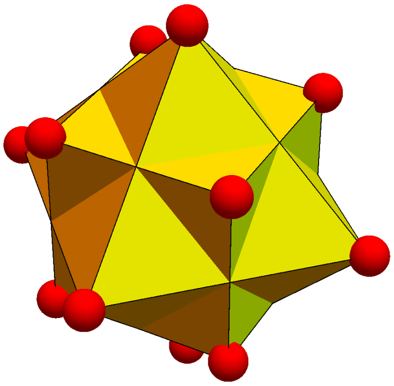
We use the Green theorem. The curl of $\vec{F}$ is constant 2. The integral $\int\int_R \text{curl}(\vec{F})\ dxdy$ is therefore 2 times the area of R which is $2 \times 5 = \boxed{10}$.

Problem 12) (10 points)

Find the flux of the vector field

$$\vec{F}(x,y,z) = [4x + z, 2y + x, 3z + x^2 + y^2]$$

through the **cube octahedron compound** E of volume 12, seen to the right.



P.S. You might be curious to know how the volume of E is computed: it is the volume of the unit cube $-1 < x < 1, -1 < y < 1, -1 < z < 1$ (which is 8) plus the volumes of 6 pyramids of base area 2 and height 1 (which gives 4). Together, this is 12.

Solution:
We use the divergence theorem. The divergence of the vector field is 9. The integral $\int\int\int_E \text{div}(\vec{F})\, dV$ is therefore 9 times the volume of E. The result is $9 \cdot 12 = 108$.

Problem 13) (10 points)

We watch people sailing on the **Charles river**. A sail is a surface parametrized by

$$\vec{r}(s,t) = [s\sin(t), s(1-s), t]$$

$0 \leq s \leq 1, 0 \leq t \leq \pi$. The surface is oriented so that the two parametrized curves

$$\vec{r}_1(t) = [\sin(t), 0, t], 0 \leq t \leq \pi$$

and

$$\vec{r}_2(t) = [0, 0, \pi - t], 0 \leq t \leq \pi$$

together form its oriented boundary. The **wind force** is given by

$$\vec{F} = \text{curl}(\vec{G})\ , \vec{G} = [z, z\sin(z^7)y, z]\ .$$

What is the flux of $\vec{F}$ through the surface S?

Solution:
We use Stokes theorem. This requires to find the line integral of $\vec{G}$ along the two paths. We have

$$\int_0^\pi [t, t\sin(t^7)0, t] \cdot [\cos(t), 0, 1]\ dt + \int_0^\pi [t, t\sin(t^7)0, \pi - t] \cdot [0, 0, -1]\ dt = -2\ .$$

This simplifies to

$$\int_0^\pi t + t\cos(t)\ dt - \int_0^\pi t\ dt = \int_0^\pi t\cos(t)\ dt = t\sin(t)|_0^\pi - \int_0^\pi \sin(t)\ dt = -2\ .$$

In the last step we have used integration by parts. The result is $\boxed{-2}$.

Problem 14) (10 points)

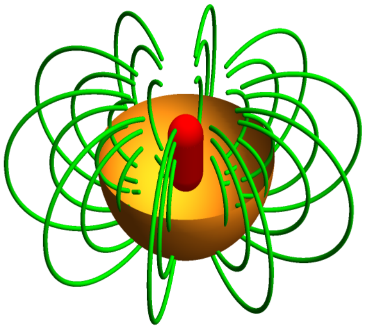
The vector field

$$\vec{A}(x,y,z) = \frac{[-y,x,0]}{(x^2+y^2+z^2)^{3/2}}$$

is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2+y^2+z^2=1, z \leq 0$ oriented downwards.



Solution:
Since we have an integral of the curl of the vector field $\vec{A}$, we use **Stokes theorem** and integrate $\vec{A}(\vec{r}(t))$ along the boundary curve $\vec{r}(t) = [\cos(t), -\sin(t), 0]$. First of all, we have $\vec{A}(\vec{r}(t)) = [\sin(t), \cos(t), 0]$. The velocity is $\vec{r}\,'(t) = [-\sin(t), \cos(t), 0]$. The integral is $\int_0^{2\pi} -1 \; dt = -2\pi$. The answer is $\boxed{-2\pi}$.