8/5/2021 PRACTICE EXAM 5 **Maths 21a, O. Knill, Summer 2021**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) No justifications are necessary

1) T F The parametrization $\vec{r}(u,v) = [v\cos(u), v, v\sin(u)]$ describes a cone.

2) T F The projection vector $\vec{P}_{\vec{w}}(\vec{v})$ of $\vec{v}$ onto $\vec{w}$ always has length smaller or equal than the length of $\vec{w}$.

3) T F The vectors $\vec{v} = [1,2,3]$ and $\vec{w} = [3,2,1]$ are parallel.

4) T F Let S is the unit sphere and $\vec{F}$ is a vector field in space satisfying $\text{div}(\vec{F}) = 0$ everywhere, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

5) T F If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) then $\int_C \vec{F} \cdot d\vec{r}(t) = 0$ for any closed curve C.

6) T F If $\vec{F}(x,y,z)$ has zero divergence everywhere in space, then $\vec{F}$ has zero curl everywhere in space.

7) T F If $\vec{F}, \vec{G}$ are two vector fields which have the same curl then $\vec{F} - \vec{G}$ is a constant vector field.

8) T F As you know, we also write $\text{grad}(f) = \nabla f$ for the gradient. In $\mathbf{R}^3$ the equation $\text{grad}(\text{curl}(\text{grad}(f))) = \vec{0}$ always holds.

9) T F The linearization of $f(x,y) = 1 + 3x$ is the function $L(x,y) = 3x$.

10) T F If $\vec{r}(t), a \leq t \leq b$ is part of a flow line of a vector field $\vec{F}(x,y,z)$ for which $|\vec{F}(x,y,z)| = 1$ at every point, then $|\int_a^b \vec{F}(\vec{r}(t))\, dt|$ is the arc length.

11) T F There is a non-constant function $f(x,y,z)$ such that $\text{grad}(f) = \text{curl}(\text{grad}(f))$.

12) T F If $\vec{F}$ is a vector field and E is the solid region $x^4 + y^4 + z^4 \leq 1$, then $\iiint_E \text{div}(\text{curl}(\vec{F}))\, dxdydz = 0$.

13) T F If the vector field $\vec{F}$ has constant divergence 1 everywhere, then the flux of $\vec{F}$ through any closed surface S is the volume of the enclosed solid.

14) T F The vector $\vec{j} \times (\vec{j} \times \vec{i})$ is the zero vector, if $\vec{i} = [1,0,0], \vec{j} = [0,1,0]$, and $\vec{k} = [0,0,1]$.

15) T F If $f(x,y)$ is maximized at (a,b) under the constraint $g = c$, then $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel.

16) T F The distance between a point P and the line L through two different points A, B is given by the formula $|\vec{PA} \cdot \vec{AB}|/|\vec{AB}|$.

17) T F There exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = \vec{F}$.

18) T F The unit tangent vector $\vec{T}(t)$ is perpendicular to the vector $\vec{T}\,'(t)$.

19) T F The vector field $\vec{F}(x,y,z) = [x, 2x, 3x]$ can not be the curl of an other vector field.

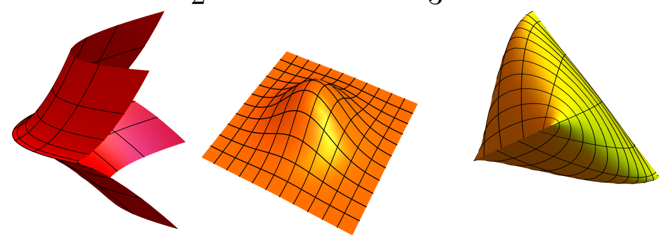
20) T F The expression $\text{div}(\text{grad}(\text{div}(\text{curl}(\text{curl}(\text{grad}(f))))))$ is a well defined function of three variables if f is a function of three variables.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match

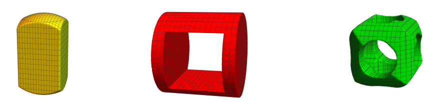
Parametrized surface $\vec{r}(u,v)$	1-3
$[u, v, \exp(-(u^2+v^2))]$	
$[\sin(v+u), \cos(u-v), \sin(u)]$	
$[v^2+u^2, u, v^3]$	

1 2 3



b) (2 points) Match the solids. There is an exact match.

Solid	a-c
$x^2 < 8, y^2+z^2 < 9$ $(x^2 > 4$ or $z^2 > 4)$	
$1 \leq x^2+z^2 \leq 4, 1 \leq x^2+y^2 \leq 4, 1 \leq y^2+z^2 \leq 4$	
$x^2+y^2+z^2 \leq 4, x^2 < 1, y^2 < 1$	

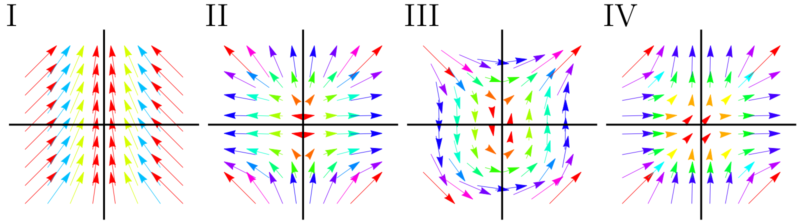


a b c

c) (2 points) The figures display vector fields. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = [x, y^2]$	
$\vec{F}(x,y) = [-x, 1]$	
$\vec{F}(x,y) = [x^2, y^2]$	
$\vec{F}(x,y) = [y^2, x]$	

I II III IV



d) (2 points) Hunt down those partial differential equations!

Equation	A-D or 0 if no match
Clairaut	
Burger	
Wave	
Laplace	

	Partial differential equations
A	$u_{tt} - u_{xx} = 0$
B	$u_{tt} + u_{xx} = 0$
C	$u_t + uu_x = 0$
D	$u_t + u_x = 0$

e) (2 points) In the second midterm, some creative nicknames were created for Fubini: we saw "Fudini", "Fumini", "The Italian guy", "The French guy". Some even called him "Clairaut". Lets see again: you task is to chose from the following names "Euler", "Hamilton", "Frenet", "Clairaut", "Fubini", "Cauchy", "Schwarz", "Laplace", "Stokes", "Gauss", "Green", "Ostogradsky", "Archimedes", "Lagrange", "Fermat".

	Name of person or pair of persons
Who showed $f_{xy} = f_{yx}$?	
Who proved $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$?	

Another joker! We have mentioned the **Maxwell equations** in the Stokes lecture class. Here are three of these equations. The fourth one is still missing. If you can write down the missing one, you can regain 2 points, possibly lost earlier in this problem.

$$\text{div}(\vec{E}) = \sigma, \text{curl}(\vec{E}) = -\vec{B}_t, \ \text{curl}(\vec{B}) = j + \vec{E}_t$$

Problem 3) (10 points) No justifications necessary

a) (5 points) Complete the formulas:

Formula	where X is
$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\|\mathbf{X}$	
$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\|\mathbf{X}$	
$\vec{P}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w})/\mathbf{X}$	
$\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ if and only if $\mathbf{X}$ is zero	
If $\vec{v} \times \vec{w} = 0$, then $\vec{v}$ and $\vec{w}$ are $\mathbf{X}$	

b) (5 points) Choose from the following words to complete the table below:

"arc length formula", "surface area formula", "chain rule", "volume", "area of parallelogram", "partial differential equation", "Fubini Theorem", "line integral", "flux integral", "Phytagoras theorem", "Al Khashi cos-formula", "vector projection", "scalar projection", "partial derivative", "unit tangent vector", "normal vector", "binormal vector"

Formula	Name of formula or rule or theorem
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$	
$\int_a^b \|\vec{r}\,'(t)\| \, dt$	
$\text{div}(\text{curl}(\vec{F})) = 0$	
$\|\vec{v} - \vec{w}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos(\theta)$	
$\vec{T}\,'(t)/\|\vec{T}\,'(t)\|$	

Problem 4) (10 points)

A recent viral video features **Otmashka Dumin-ina** sky diving from the **Aizhai bridge** in China. The clip became famous because she first had to get rid of police officers who were trying to prevent her jump, wink them good-by and then jump. There is some wind. The acceleration is

$$\vec{r}\,''(t) = [1+t, t, -10] \ .$$

Source: Youtube

a) (5 points) Write down $\vec{r}(t)$ and find the position $\vec{r}(2)$, if $\vec{r}(0) = [0, 0, 300]$ and $\vec{r}\,'(0) = [1, 0, 0]$.

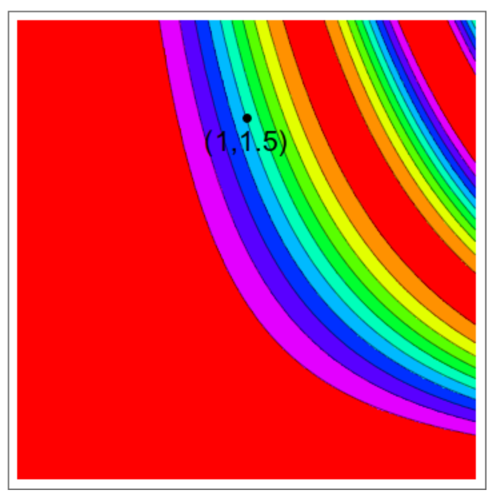
b) (5 points) Find the curvature

$$\frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

of the path at $\vec{r}(2)$.

Problem 5) (10 points)

In this problem, we work with the function $f(x, y) = \cos(x^2 y)$.



a) (4 points) Estimate $f(1.001, \pi/2 - 0.0001)$ using linear approximation.

b) (3 points) Find the tangent plane to the surface $g(x, y, z) = \cos(x^2 y) - z^2 + 2z = 0$ at $(1, \pi/2, 2)$.

c) (3 points) Find the direction = unit vector in which the directional derivative $D_{\vec{v}} f(1, \pi/2)$ is maximal.

Problem 6) (10 points)

While searching for a geometry problem, Oliver sat at the **Harvard square coffee-shop** just in front of spherical lamps located at

$$A = (1,2,4), B = (2,2,3), C = (3,1,5) \ .$$

Foto by Oliver taken on August 2, 2016

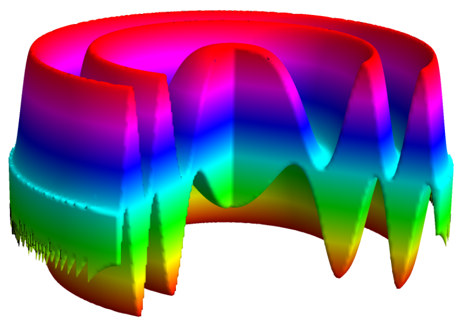
Well, here it is, the geometry problem:

a) (3 points) Find the area of the triangle ABC.

b) (3 points) Find the equation of the plane through A,B and C.

b) (4 points) Find the distance of C to the line through the points A and B.

Problem 7) (10 points)



Find the volume of the solid given by the inequalities

$$x^2 + y^2 \leq 16$$

$$\sin(x^2 + y^2) \leq z \leq 2 + \cos(x^2 + y^2)$$

$$x \geq 0 \quad \text{or} \quad y \geq 0 \ .$$

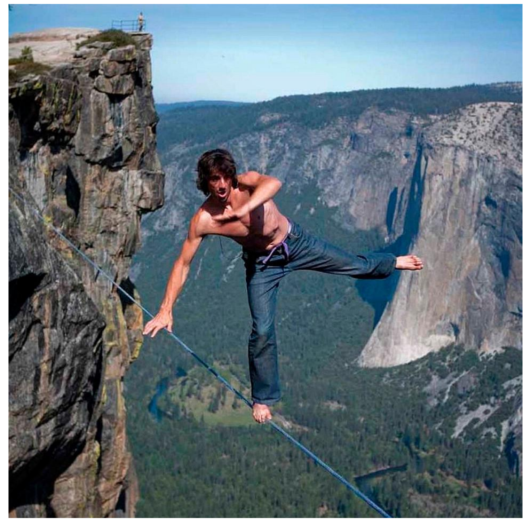
As the picture illustrates, the last condition means that the points (x, y, z) for which $x < 0, y < 0$ are excluded.

Problem 8) (10 points)

We like it **"extreme"**. Maximize the function

$$f(x,y) = x^{1000} - 1000x + y^{200} - 100y^2 \; .$$

To do so, find all critical points and classify them using the second derivative test. This is an interesting example, as computer algebra systems have trouble writing down the solutions (they look for solutions in the complex). But we can find the real ones!



Dean Potter solo walking at Taft Point in Yosemite

Problem 9) (10 points)

a) (5 points) On the other extreme side, here is one of the simplest Lagrange problems: extremize

$$f(x,y) = x^2 - y$$

on the curve

$$g(x,y) = y - x - 1 = 0$$

using the Lagrange method.

b) (5 points) **Economists** also solve Lagrange problems by finding the critical points of $F(x,y,z) = f(x,y) - zg(x,y)$ of three variables. (z plays the role of the Lagrange multiplier.) Write down $F(x,y,z)$ for the example given in a) and find its critical points $\nabla F(x,y,z) = 0$. You should get the same x, y values as in a).

Problem 10) (10 points)

Three days ago, on August 1, 2021, Lukas Irmler walked over a rope over the **Rheinfalls** in Switzerland. There is a force field $\vec{F}$ present which consists part of the gravitational force and part by the wind forces:

$$\vec{F}(x,y,z) = [\sin(x), \cos(y), -10+z] \ .$$

The path is given by $\vec{r}(t) = [5t, t, 30 - \sin(t)/10]$, where $0 \leq t \leq \pi$. Compute the work

$$\int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$$

done by Lukas during this stunt.

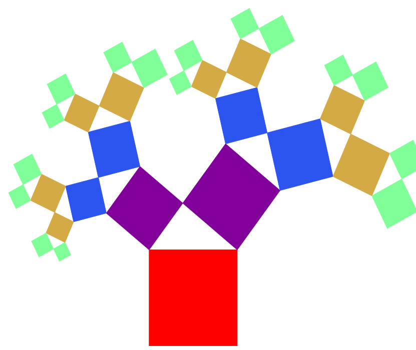
Source: Schaffhauser Nachrichten, August 2, 2021

Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = [-y + x^8, x - y^9]$$

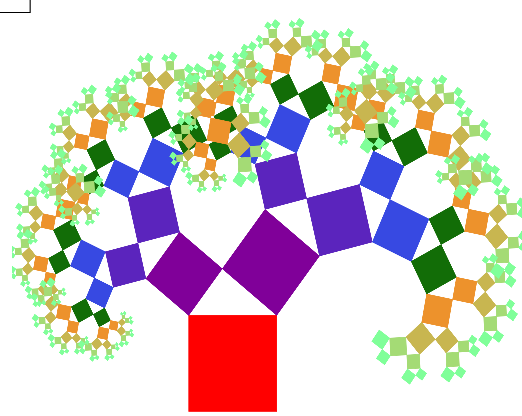
along the boundary C of the generation 4 **Pythagoras tree** shown in the picture. The curve C traces each of the 31 square boundaries counter clockwise. You can use the Pythagoras tree theorem mentioned below. We also included the proof of that theorem evenso you do not need to read the proof in order to solve the problem.



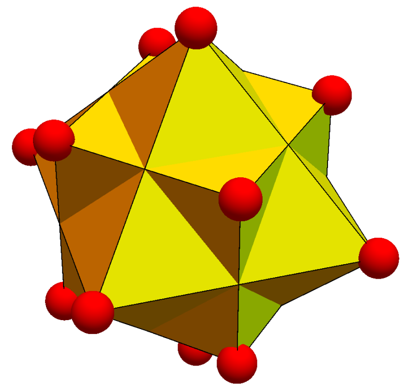
Pythagoras tree theorem:

The generation n Pythagorean tree has area $n+1$.

Proof: in each generation, new squares are added along a right angle triangle. The 0'th generation is a square of area $c^2 = 1$. The first generation tree got two new squares of side length a, b which by **Pythagoras** together have area $a^2 + b^2 = c^2 = 1$. Now repeat the construction. In generation 2, we have added 4 new squares which together have area 1 so that the tree now has area 3. In generation 3, we have added 8 squares of total area 1 so that the generation tree has area 4. Etc. Etc. The picture to the right shows generation 7. Its area of all its (partly overlapping) leafs is 8.



Problem 12) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = [4x+z, 2y+x, 3z+x^2+y^2]$$

through the **cube octahedron compound** E of volume 12, seen to the right.

P.S. You might be curious to know how the volume of E is computed: it is the volume of the unit cube $-1 < x < 1, -1 < y < 1, -1 < z < 1$ (which is 8) plus the volumes of 6 pyramids of base area 2 and height 1 (which gives 4). Together, this is 12.

Problem 13) (10 points)

We watch people sailing on the **Charles river**. A sail is a surface parametrized by

$$\vec{r}(s,t) = [s\sin(t), s(1-s), t]$$

$0 \leq s \leq 1, 0 \leq t \leq \pi$. The surface is oriented so that the two parametrized curves

$$\vec{r}_1(t) = [\sin(t), 0, t], 0 \leq t \leq \pi$$

and

$$\vec{r}_2(t) = [0, 0, \pi - t], 0 \leq t \leq \pi$$

together form its oriented boundary. The **wind force** is given by

$$\vec{F} = \text{curl}(\vec{G}) \ , \vec{G} = [z, z\sin(z^7)y, z] \ .$$

What is the flux of $\vec{F}$ through the surface S?

Problem 14) (10 points)

The vector field

$$\vec{A}(x,y,z) = \frac{[-y,x,0]}{(x^2+y^2+z^2)^{3/2}}$$

is called the **vector potential** of the magnetic field

$$\vec{B} = \text{curl}(\vec{A}).$$

The picture shows some flow lines of this **magnetic dipole field** $\vec{B}$. Find the flux of $\vec{B}$ through the lower half sphere $x^2+y^2+z^2=1, z \leq 0$ oriented downwards.