8/5/2021 PRACTICE EXAM 3 **Maths 21a, O. Knill, Summer 2021**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

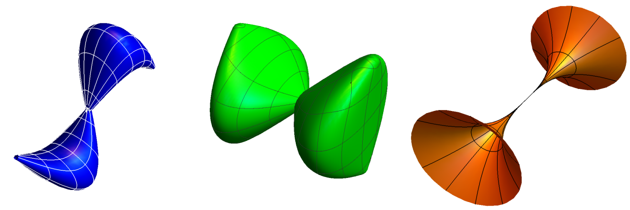
1)	T	F	The distance d between a point P and a line through two distinct points A, B is given by $d = |\vec{AP} \times \vec{AB}|/|\vec{AB}|$.
2)	T	F	The set of all points in three dimensional space satisfying $x^2 + y^2 = 1$ define a circle.
3)	T	F	The surface area of a surface parametrized by $\vec{r}(u,v)$ on a parameter region R is given by $\int\int_R |r_u \times r_v| \, dudv$.
4)	T	F	Given the curve $\vec{r}_1(t) = [\cos(3t), \sin(3t), \cos(5t)]$ and the curve $\vec{r}_2(t) = [\cos(6t), \sin(6t), \cos(10t)]$. Then the TNB-frame of of $\vec{r}_1$ at $t = 0$ is the same as the TNB frame of $\vec{r}_2$ at $t = 0$.
5)	T	F	The curve $\vec{r}(t)$ has at the point $P = \vec{r}(0)$ the curvature 2. Then, the curve $\vec{r}(2t)$ has at the same point P the curvature 4.
6)	T	F	For any vector field $\vec{F} = [P, Q, R]$ the identity $\text{grad}(\text{div}(\text{curl}(\vec{F}))) = \text{curl}(\text{grad}(\text{div}(\vec{F})))$ holds.
7)	T	F	For any closed curve $r(\vec{t})$ with $t \in [0,1]$ of positive length for which $|\vec{r}(t)| = 1$ for all t there is a point, where the curvature is non-zero.
8)	T	F	If $f(x,y)$ is non-zero, the integral $\iint_R f(x,y) \, dxdy$ is a volume and so positive.
9)	T	F	The double integral $\int_0^1 \int_y^1 2e^{-x^2} \, dxdy$ evaluates to $1 - 1/e$.
10)	T	F	If $\vec{v} = \vec{r}(0) = \vec{r}\,'(0)$ is non-zero and $\vec{r}\,''(t)$ is always parallel to $\vec{v}$, then $\vec{r}(t)$ moves on part of a line through the origin 0).
11)	T	F	Given a surface S with boundary C with compatible orientations, then Stokes theorem implies $\int_C |\vec{r}\,'(t)| \, dt = \int_S |\vec{r}_u \times \vec{r}_v| \, dudv$.
12)	T	F	The equation $x^2 - (z-1)^2 + y^2 + 2y = -1$ represents a two-sheeted hyperboloid.
13)	T	F	Any incompressible and irrotational vector field $\vec{F} = [P, Q, R]$ is built by linear functions P, Q, R.
14)	T	F	The curvature of the curve $\vec{r}(t) = [\cos(t^2)/3, \sin(t^2)/3]$ at $t = 1$ is everywhere equal to 3.
15)	T	F	The distance $d(S_1, S_2)$ between spheres S_1, S_2 is defined as the minimum of all $d(P_1, P_2)$, where $P_1 \in S_1$ and $P_2 \in S_2$. Given 3 spheres S_1, S_2, S_3, then $d(S_1, S_2) + d(S_2, S_3) \geq d(S_1, S_3)$ holds.
16)	T	F	If $\vec{F}$ and $\vec{G}$ are vector fields for which the divergence is 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field.
17)	T	F	Let S is an ellipsoid, oriented outwards and $\vec{F}$ is a vector field in space which is the curl of an other vector field, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.
18)	T	F	The parametrization $\vec{r}(u,v) = [u^3, 1, v^3]$ describes a plane.
19)	T	F	The flux of the curl of the field $\vec{F} = [0, x, 0]$ through the unit disc $\{(x,y,z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to π.
20)	T	F	The flux of the vector field $\vec{F} = [x, 0, 0]$ through the unit disc $\{(x,y,z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to the area of the disc.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(s,t)$	A-C
$[\cos(t)s^3, 2s, \sin(t)s^3]$	
$[\sin(t), \sin(2t)\sin(s), \sin(t)\cos(s)]$	
$[\cos(t)\sin(s), s, (2+\sin(t))\sin(s)]$	

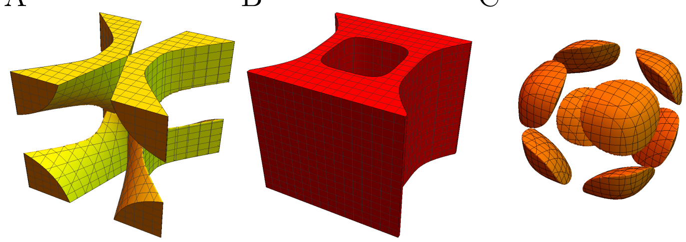
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^4 - y^4 < 4, x^4 + y^4 > 1$	
$x^4 z^4 > y^4, x^4 - y^4 < 1$	
$x^4 + y^4 + z^4 < 16, \|xyz\| > 1$	

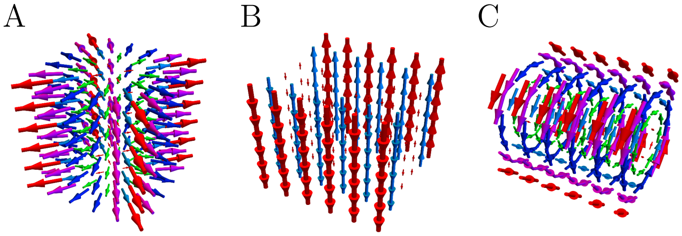
A B C



c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F}(x,y,z) = [\sin(x), \sin(y), 0]$	
$\vec{F}(x,y,z) = [0, 0, \sin(y)]$	
$\vec{F}(x,y,z) = [0, -\sin(z), \sin(y)]$	

A B C



d) (2 points) Recognize partial differential equations!

Equation	A-F
Laplace	
Burger	
Wave	

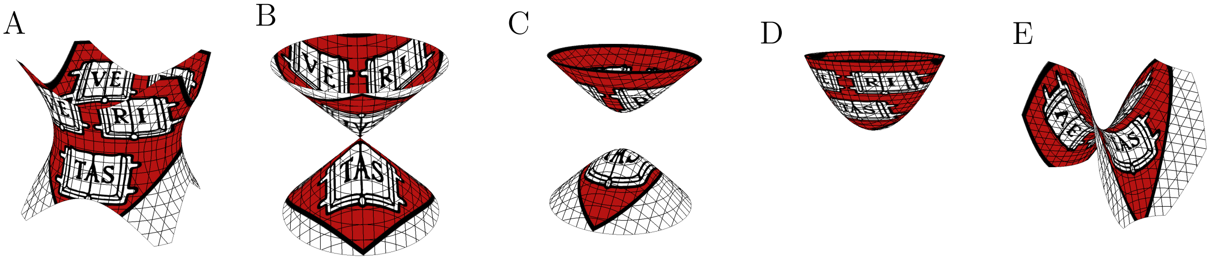
	PDE
A	$u_t^2 + u_x^2 = 1$
B	$u_t = u_x$
C	$u_{tt} = -u_{xx}$

	PDE
D	$u_t = u_x$
E	$u_{tt} = u_{xx}$
F	$u_t = -uu_x$

e) (2 points) Find the true quadrics! Veritas!

	Enter two letters from A-E each
Which ones are paraboloids?	
Which ones are hyperboloids?	

A B C D E



Problem 3) (10 points) No justifications necessary

a) (5 points) Enter either equal, or inequality signs. In case, there should be no relation in general, enter the "not-equal sign" $\neq$. The vectors $\vec{v}, \vec{w}$ are vectors in three dimensional space and α is the angle between the two vectors. The letters $\vec{T}, \vec{N}, \vec{B}$ denote the TNB frame in a space curve.

Formula 1	Enter $=, \leq, \geq, \neq,$	Formula 2
$\lvert\vec{N}\rvert$		$\lvert\vec{B}\rvert$
$\lvert\vec{v} \cdot (\vec{v} \times \vec{w})\rvert$		$\lvert\vec{v} \times \vec{v}\rvert$
$\lvert\vec{v} - \vec{w}\rvert$		$\lvert\vec{v}\rvert + \lvert\vec{w}\rvert$
$\lvert\vec{v} \cdot \vec{w}\rvert$		$\lvert\vec{v}\rvert\ \lvert\vec{w}\rvert$
$\lvert\vec{v} \cdot \vec{w}\rvert$		$\lvert\vec{v}\rvert\lvert\vec{w}\rvert\cos(\alpha)$

This space is for relaxation purposes only. Take some rest.

b) (5 points) We list here **12 of the most important theorems in multivariable calculus** and identify what they are about, then vote which is our favorate. The statement "involves vector fields" means that the statement of the theorem involves a vector field. The statement "involves integrals" means that when writing down the theorem, there appears at least one integral.

Number	Theorem	Involves vector fields	Involves integrals
1)	Pythagoras theorem		
2)	Al Khashi theorem		
3)	Cauchy-Schwarz theorem		
4)	Clairaut theorem		
5)	Fubini theorem		
6)	Gradient theorem		
7)	Second derivative test		
8)	Lagrange theorem		
9)	Fundamental theorem of line integrals		
10)	Greens theorem		
11)	Stokes theorem		
12)	Divergence theorem		

Which is your favorite? (Your choice is not affecting your score of this problem).

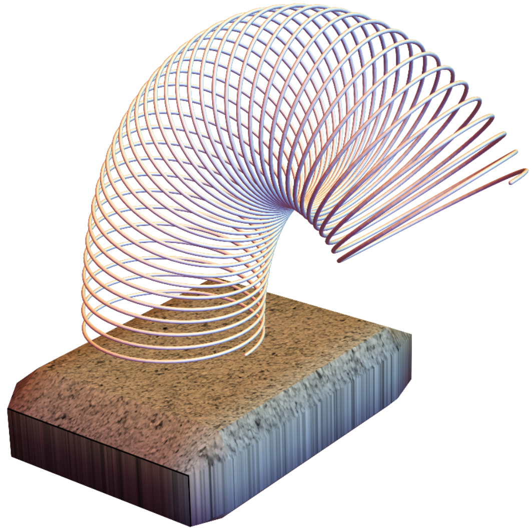
My favorite theorem is theorem number [ ].

Problem 4) (10 points)

It is fun to play with the **slinky** as it can climb down stairs while the arc length of the slinky stays the same. We compute here the motion $\vec{r}(t)$ of the top slinky part by giving the acceleration

$$\vec{r}\,''(t) = \begin{bmatrix} -100^2\cos(100t) \\ -100^2\sin(100t) \\ -1 \end{bmatrix}$$

and the initial position $\vec{r}(0) = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and initial velocity $\vec{r}\,'(0) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$. Find the path $\vec{r}(t)$.



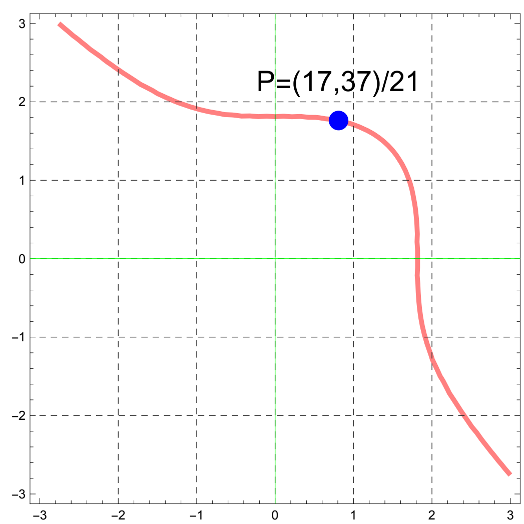
Problem 5) (10 points)

The book **"Magnificant Mistakes in Mathematics"** reports that Legendre conjectured in a book of 1794 that there are no rational solutions of the equation

$$x^3 + y^3 = 6 \ .$$

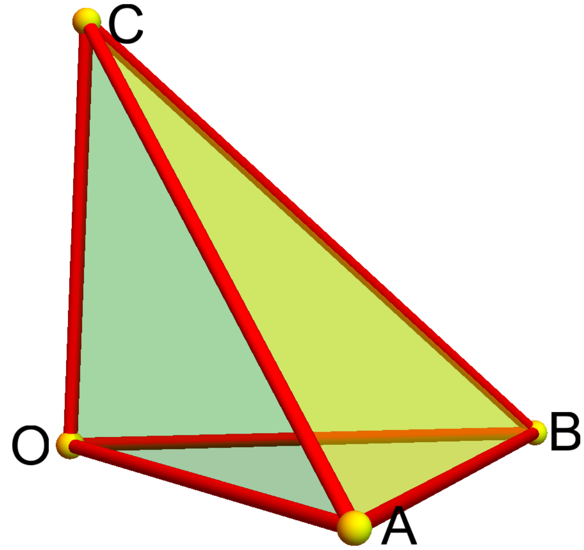
Henry Ernest Dudeny proved this wrong by stating $(17/21)^3 + (37/21)^3 = 6$.

a) (3 points) Find a nonzero vector $[a, b]$ perpendicular to the curve at $(17/21, 37/21)$.
b) (4 points) Find the tangent line to the curve at $(17/21, 37/21)$.
c) (3 points) Find the linearization $L(x, y)$ at that point.



Problem 6) (10 points)

a) (2 points) Find the equation $ax + by + cz = d$ of the plane through $A = (2, 0, 0), B = (0, 3, 0), C = (0, 0, 5)$.
b) (2 points) Determine its distance h of the plane to the origin $O = (0, 0, 0)$.
c) (2 points) Compute the area T of the triangle ABC.
d) (2 points) What is the volume $V = T \cdot h/3$ of the tetrahedron?
e) (2 points) The **3D Pythagoras theorem** states that the square of the area of ABC is the sum of the squares of the areas of the triangles OAB, OBC and OCA (which are each half of a rectangle). Check this in the current situation.



Problem 7) (10 points)

We design a perfume line "phoenix" with the slogan **"Its bold scent embodies the maveric spirit. Be reborn everyday!"**

The bottle E is a solid given by the region

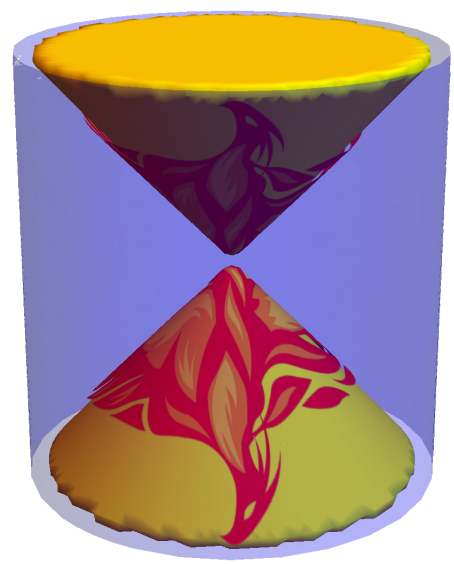
$$x^2 + y^2 \geq z^2,\ x^2 + y^2 \leq 1,\ -1 \leq z \leq 1$$

which is the complement of a cone in a cylinder. The perfume density is $x^2 + y^2$. Find the total mass

$$\int\int\int_E x^2 + y^2 \, dzdxdy \, .$$

of the perfume.

Remark: This slogan is plagiarized from an actual three letter cosmetic line.



Problem 8) (10 points)

We want to build a station on our **moon base** and find a place where the terrain is optimal. Given a height function $g(x,y) = xy + x^2 + y^2$, we can look at the place where the magnitude of the gradient $f(x,y) = |\nabla g|^2 = g_x^2 + g_y^2$ is extremal.

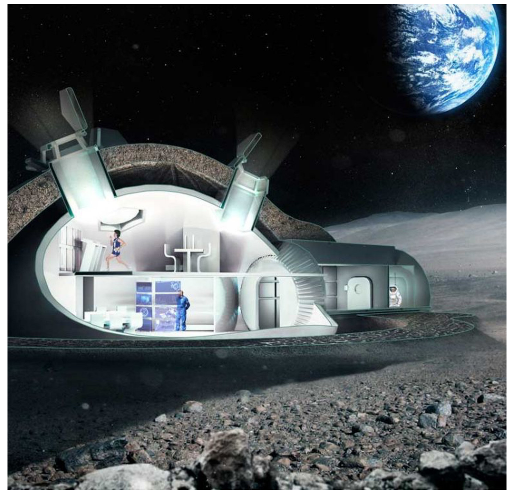


Image credit: ESA

a) (8 points) Classify the critical points of this function

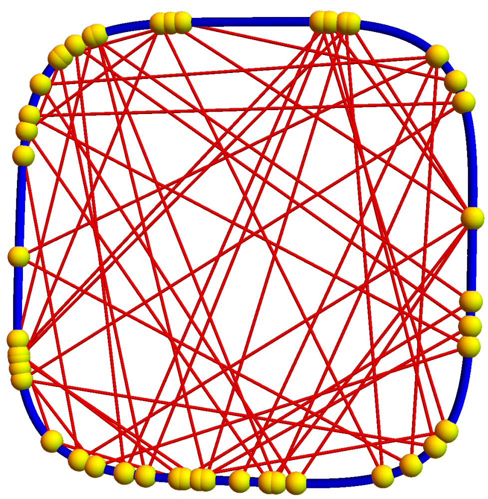
$$f(x,y) = 5x^2 + 8xy + 5y^2$$

using the second derivative test.

b) (2 points) You should have found either a maximum or minimum or a saddle point. Is there a global maximum or global minimum among them?

Problem 9) (10 points)

The l^p **billiard table** for $p = 4$ is given by $g(x,y) = x^4 + y^4 = 1$. Finding trajectories amounts of extremizing the length of the trajectory. The picture to the right shows a few bounces of this billiard. It is an open question whether it shows chaos on a set of initial conditions which have positive area.
We solve the simpler problem to find the points on that curve, where the distance to the origin is minimal. To do so, we extremize the function $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = 1$. Find the maxima and minima using the Lagrange multiplier method.



Problem 10) (10 points)

According to **traditional Chinese medicine**, the human body contains 14 meridians. Each meridian is a pathway for life-energy known as "Qi". We don't know how and why they work but lets model them with vector fields and assign the **Qi energy** $\int_C \vec{F} \cdot d\vec{r}$, where C the meridian between acupuncture point $A = (1, 2, 1) = \vec{r}(0)$ and $B = (2, 2, 10) = \vec{r}(1)$. The spleen vector field is $\vec{F}$, the meridian C is parametrized by $\vec{r}(t)$ with $0 \leq t \leq 1$. The formulas are

$$\vec{F}(x,y,z) = \begin{bmatrix} x^3 + y \\ y^3 + x \\ z^5 \end{bmatrix}, \quad \vec{r}(t) = \begin{bmatrix} 1 + t \\ 2 + \sin(15\pi t) \\ 9t + \cos(2\pi t^2) \end{bmatrix}$$

Find the Qi energy!

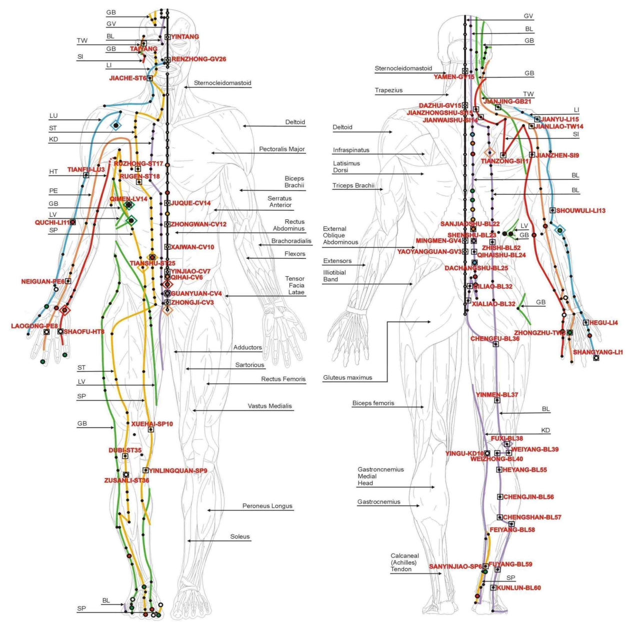


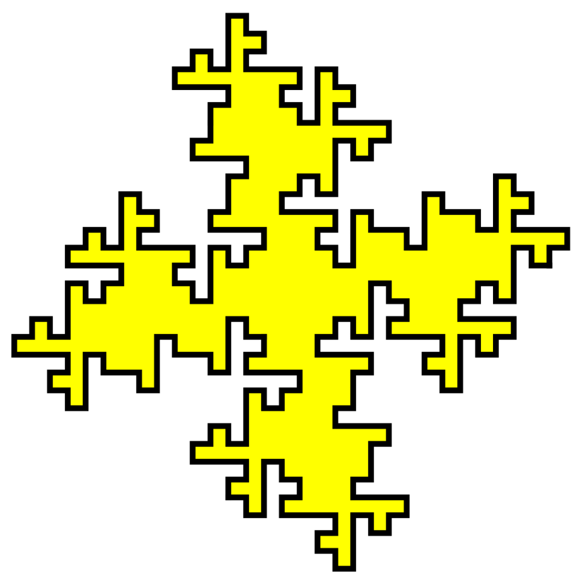
Image source: Wikipedia

Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = \begin{bmatrix} x + y \\ 3x + 3y^2 \end{bmatrix}$$

along the boundary C of the **Koch island** shown to the right. The curve is oriented counter clockwise and encloses a region G which has 289 unit squares. The length of the curve C is 324.



Remark. The curve was computed using an **L-system** construction, given by rules "$F + F + F + F$", "F" $\rightarrow$ "$F - F + F + FFF - F - F + F$" which is not explained further here. You have all the information needed to solve the problem.

Problem 12) (10 points)

A famous photograph shows **Nikola Tesla** reading within a firework of sparks produced by **Tesla coils**. The electric field is given

$$\vec{F}(x,y,z) = \begin{bmatrix} x^3 \\ y^3 \\ z^3 \end{bmatrix} .$$

Find the flux $\int\int_S \vec{F} \cdot d\vec{S}$ of the vector field through outwards oriented sphere

$$S : x^2 + y^2 + z^2 = 9 .$$

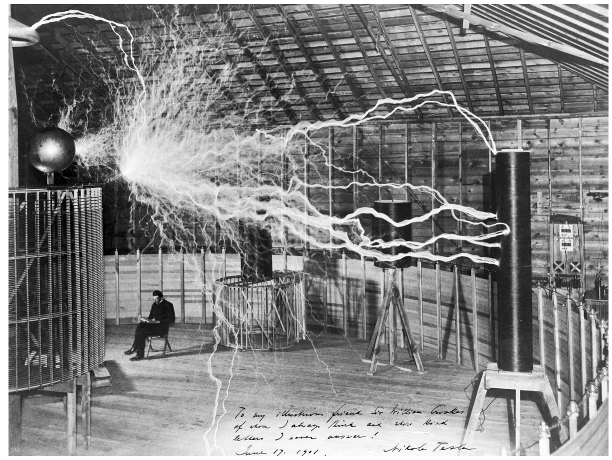


Image source: TeslasAutobiography

Problem 13) (10 points)

Inspired by electronically connected cars like Tesla's Model 3, we build a **high tech triathlon bike** handle S with integrated electronics. This surface S is given by

$$\vec{r}(t,s) = \begin{bmatrix} (9 + (2 + \sin(5s)/2)\cos(t))\sin(s) \\ (9 + (2 + \sin(5s)/2)\cos(t))\cos(s) \\ (9 + \sin(5s)/2)\sin(t)/3 \end{bmatrix}$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$. The built-in phone measures the flux $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$ of the curl of the earths magnetic field

$$\vec{F} = \begin{bmatrix} z \\ 0 \\ y \end{bmatrix}$$

through the surface S which is oriented outwards and has two boundary parts: one,when $s = 0$ and one when $s = \pi$. Compute this flux.