Checklist III

Triple Integrals

- $\iiint_R f(x,y,z)\ dzdydx$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx$ integral over rectangular box
- $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\ dzdydx$ type I region
- $\iiint_R f(r,\theta,z)\ \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
- $\iiint_R f(\rho,\theta,z)\ \boxed{\rho^2 \sin(\phi)}\ dzdrd\theta$ integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\ dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\ dxdydz$ Fubini
- $V = \iiint_E \boxed{1}\ dzdydx$ volume of solid E

Line Integrals

- $\vec{F}(x,y) = [P(x,y), Q(x,y)], \vec{F}(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]$ vector field.
- $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ line integral
- $\vec{F}(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals

- FTL: $\vec{F}(x,y) = \nabla f(x,y)$, $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$
- closed loop property $\int_C \vec{F}\ d\vec{r} = 0$, for all closed curves C
- always equivalent: closed loop property, path independence and gradient field
- Clairaut test $\text{curl}(\vec{F}) \neq 0$ assures $\vec{F}$ is not a gradient field

Green's Theorem

- $\vec{F}(x,y) = [P,Q]$, curl in two dimensions: $\text{curl}(\vec{F}) = Q_x - P_y$
- Green's theorem: C boundary of R, then $\int_C \vec{F} \cdot dr = \int\int_R \text{curl}(\vec{F})\ dxdy$
- Take $\vec{F} = [-y, 0]$ or $\vec{F} = [0, x]$ to get area
- Green's theorem is useful to compute difficult line integrals or difficult 2D integrals

Flux integrals

- $\vec{F}(x,y,z)$ vector field, $S = \vec{r}(R)$ parametrized surface
- $\vec{r}_u \times \vec{r}_v$ normal vector, $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ unit normal vector
- $\vec{r}_u \times \vec{r}_v\ dudv = d\vec{S} = \vec{n}\ dS$ normal surface element
- $\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv$ flux integral

Stokes Theorem

- $\vec{F}(x,y,z) = [P,Q,R]$, $\text{curl}([P,Q,R]) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = \nabla \times \vec{F}$
- Stokes's theorem: C boundary of surface S, then $\int_C \vec{F} \cdot d\vec{r} = \int\int_S \text{curl}(\vec{F}) \cdot d\vec{S}$
- Stokes theorem allows to compute difficult flux integrals or difficult line integrals

Grad Curl Div

- $\nabla = [\partial_x, \partial_y, \partial_z]$, $\vec{F} = \nabla f$, $\text{curl}(\vec{F}) = \nabla \times \vec{F}$, $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$
- $\text{div}(\text{curl}(\vec{F})) = 0$ and $\text{curl}(\text{grad}(\vec{f})) = \vec{0}$
- $\text{div}(\text{grad}(f)) = \Delta f$ Laplacian
- incompressible = divergence free field: $\text{div}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{curl}(\vec{G})$
- irrotational = $\text{curl}(\vec{F}) = 0$ everywhere. Implies $\vec{F} = \text{grad}(f)$

Divergence Theorem

- $\text{div}([P,Q,R]) = P_x + Q_y + R_z = \nabla \cdot \vec{F}$
- divergence theorem: solid E, boundary S then $\int\int_S \vec{F} \cdot dS = \int\int\int_E \text{div}(\vec{F})\, dV$
- the divergence theorem allows to compute difficult flux integrals or difficult 3D integrals

Some topology

- interior of a region D: points in D for which small neighborhood is still in D
- boundary of a curve: the end points of the curve if they exist
- boundary of a surface S points on surface belonging to parameters (u,v) not in the interior of the parameter domain
- boundary of a solid D: the surfaces which bound the solid, points in the solid which are not in the interior of D
- closed surface: a surface without boundary like for example the sphere
- closed curve: a curve with no boundary like for example a knot

Integration overview

- Line integral: $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$
- Flux integral: $\int\int_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dudv$

Theorems and Derivatives overview

$$1$$
$$1 \xrightarrow{\text{grad}} 1$$
$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$
$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

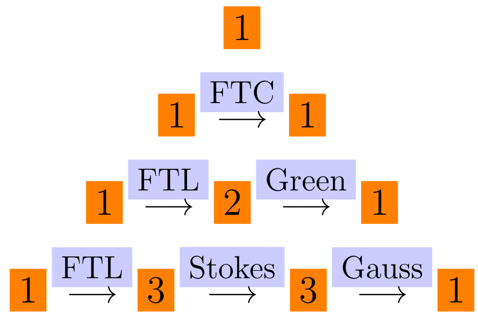
$$1$$
$$1 \xrightarrow{\text{FTC}} 1$$
$$1 \xrightarrow{\text{FTL}} 2 \xrightarrow{\text{Green}} 1$$
$$1 \xrightarrow{\text{FTL}} 3 \xrightarrow{\text{Stokes}} 3 \xrightarrow{\text{Gauss}} 1$$

Integral theorems overview

The integral theorems are summarized by **the fundamental theorem of multivariable Calculus**:

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.



Fundamental theorem of line integrals: if C is an oriented curve with boundary $\{A, B\}$ and f is a function, then

$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

Remarks.
1) For closed curves, $\int_C \nabla f \cdot \vec{dr} = 0$.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then $\int_A^B \vec{F} \ \vec{dr}$ does not depend on the path connecting A with B.
3) The theorem justifies the name **conservative** for gradient vector fields.
4) The term “potential” was coined by George Green who lived from 1793-1841.

1 Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = [\cos(5t), \sin(2t), t^2]$ from $t = 0$ to $t = 2\pi$.
Solution. $\vec{r}(0) = [1, 0, 0]$ and $\vec{r}(2\pi) = [1, 0, 4\pi^2]$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f \ \vec{dr} = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 4\pi^2$.

Green’s theorem. If R is a region with oriented boundary C and $\vec{F}$ is a vector field, then

$$\int\int_R \text{curl}(\vec{F}) \ dxdy = \int_C \vec{F} \cdot \vec{dr} \ .$$

Remarks.
1) The curve is oriented in such a way that the region is to the left.
2) The boundary of the curve can consist of piecewise smooth pieces.
3) If $C : t \mapsto \vec{r}(t) = [x(t), y(t)]$, the line integral is $\int_a^b [P(x(t), y(t)), Q(x(t), y(t))] \cdot [x'(t), y'(t)] \ dt$.
4) Green’s theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
5) If $\text{curl}(\vec{F}) = 0$ everywhere in the plane, then the field is a gradient field.
7) $\vec{F}(x, y) = [0, x]$ we get an **area formula**.
8) Don’t mix up Green with Stokes. Green is a theorem in “flatland”, where $\vec{F}$ has two components. and $\text{curl}(\vec{F})$ is a scalar.

2 Find the line integral of the vector field $\vec{F}(x,y) = [x^4 + \sin(x) + y, x + y^3]$ along the path $\vec{r}(t) = [\cos(t), 5\sin(t) + \log(1 + \sin(t))]$ with $t \in [0,\pi]$.
Solution. Since $\text{curl}(\vec{F}) = 0$ we can take the simpler path $\vec{r}(t) = [-t, 0], -1 \le t \le 1$, which has velocity $\vec{r}\,'(t) = [-1, 0]$. The line integral is $\int_{-1}^{1} [t^4 - \sin(t), -t] \cdot [-1, 0]\, dt = -t^5/5|_{-1}^{1} = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \,.$$

Remarks.
1) Stokes theorem implies Green's theorem: if $\vec{F} = [P, Q, 0]$ is z-independent and the surface S is contained in the xy-plane.
2) The orientation of C is such that if you walk along C with head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, of S, then S to your left.
3) Stokes theorem was also found by André Ampère (1775-1836) in 1825. George Stokes (1819-1903) gave it as a multivariable exam problem.
4) The flux of the curl of $\vec{F}$ through S only depends on the boundary C of S.
5) The flux of the curl through a closed surface is zero.

3 Find the line integral of $\vec{F}(x,y,z) = [x^3 + xy, y, z]$ along the polygonal path C connecting $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.
Solution. The path C bounds a surface $S : \vec{r}(u,v) = [u, v, 0]$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(\vec{F})(x,y,z) = [0, 0, -x]$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = [1,0,0] \times [0,1,0] = [0,0,1]$ so that $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 [0, 0, -u] \cdot [0,0,1]\, dudv = \int_0^2 \int_0^1 -u\, dudv = -2$.

Divergence theorem: If S is the oriented boundary of a solid region E in space and $\vec{F}$ is a vector field, then

$$\int\int\int_B \text{div}(\vec{F})\, dV = \int\int_S \vec{F} \cdot d\vec{S} \,.$$

Remarks.
1) The surface S is oriented so that the normal vector points away from E.
2) The divergence theorem is also called **Gauss theorem**.
3) It can be helpful to determine the flux of vector fields through surfaces.
4) The theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
5) For divergence-free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

4 Compute the flux of the vector field $\vec{F}(x,y,z) = [-x, y, z^2]$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\, dxdydz = (3-0)(2-(-1))(4-1) = 27$.

Integral Theorems Overview

INTEGRATION.

Line integral:	$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$
Flux integral:	$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v\ dudv$
Double integral:	$\int\int_R f(x,y)\ dxdy.$
Triple integral:	$\int\int\int_E f(x,y,z)\ dxdydz.$

Area $\int\int_R 1\ dA = \int\int_R 1\ dxdy$
Arc length $\int_a^b |\vec{r}\,'(t)|\ dt$
Surface area $\int\int |\vec{r}_u \times \vec{r}_v|\ dudv$
Volume $\int\int\int_B 1\ dxdydz$

DIFFERENTIATION.

Velocity:	$\vec{r}\,'(t) = d/dt\vec{r}(t).$
Partial derivative:	$f_x(x,y,z).$
Gradient:	$\text{grad}(f) = [f_x, f_y, f_z]$
Curl in 2D:	$\text{curl}([P,Q]) = Q_x - P_y$
Curl in 3D:	$\text{curl}[P,Q,R] = [R_y - Q_z, P_z - R_x, Q_x - P_y]$
Div:	$\text{div}(\vec{F}) = \text{div}[P,Q,R] = P_x + Q_y + R_z.$

IDENTITIES.

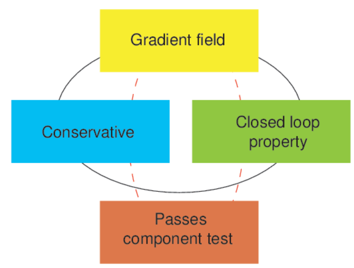
$\text{div}(\text{curl}(\vec{F})) = 0$
$\text{curl}(\text{grad}(f)) = \vec{0}$
$\text{div}(\text{grad}(f)) = \Delta f.$

JARGON.

$\text{div}(\vec{F}) = 0$ incompressible
$\text{curl}(\vec{F}) = \vec{0}$ irrotational

CONSERVATIVE FIELDS:

Gradient fields: $\vec{F} = \text{grad}(f)$.
Closed curve property: $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.
Conservative: C_1, C_2 paths from A to B, then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$.
Mixed derivative test: $\text{curl}(F) = 0$ if $\vec{F} = \nabla f$.
If $\vec{F}$ is irrotational, then $\vec{F} = \nabla f$ in simply connected R.



TOPOLOGY.
Interior of region D: points which have a neighborhood contained in D.
Boundary of curve: endpoints. **Boundary** of surface: curves. **Boundary** of solid: surfaces.
Simply connected: a closed curve in R can be deformed inside R to a point.
Closed curve Curve without boundary.
Closed surface surface without boundary.

LINE INTEGRAL THEOREM. If $C : \vec{r}(t) = [x(t), y(t), z(t)],\ t \in [a,b]$ is a curve and $f(x,y,z)$ is a function.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

GREEN'S THEOREM. If R is a region with boundary C and $\vec{F} = [P,Q]$ is a vector field, then

$$\int\int_R \text{curl}(F)\ dxdy = \int_C \vec{F} \cdot d\vec{r}$$

STOKES THEOREM. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

DIVERGENCE THEOREM. If S is the boundary of a solid E in space with boundary surface S and $\vec{F}$ is a vector field, then

$$\iiint_E \operatorname{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

GENERAL STOKES. All theorems are of the form

$$\int_G dF = \int_{\delta G} F$$

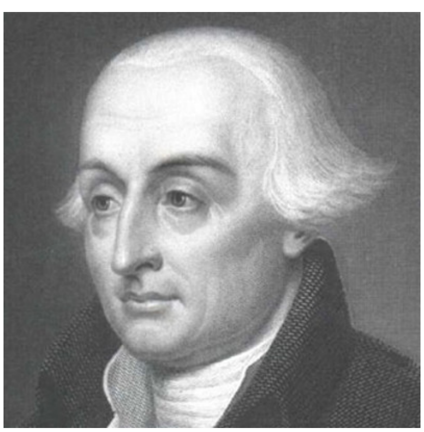
where dF is the derivative of F and δG is the oriented boundary of G.

George Gabriel Stokes

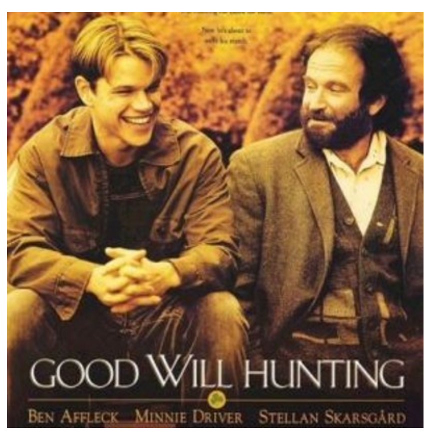
Mikhail Ostrogradsky

Carl Friedrich Gauss

André-Marie Ampère



Joseph Louis Lagrange



George Green

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