Calculus in four dimensions

Four dimensions

25.1. Calculus in four dimensions is used in **relativity**, where time t is the fourth variable. A point (x, y, z, t) defines also a **quaternion** $ix+jy+kz+t$. The Quaternions $\mathbb{H}$ are the largest real associative normed division algebra, the others being the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. The 8-dimensional octonions $\mathbb{O}$ are no more associative. The set of all lines through the origin in $\mathbb{R}^4$ is the three dimensional projective space $\mathbb{RP}^3$ which is important in **computer vision**, as one can do linear computations in $\mathbb{R}^4$. This is useful for example when figuring out the position of an object from several pictures, meaning to solve the structure from motion problem. In mechanics, higher dimensional spaces appear. A **double pendulum** state for example is given by two angles α, β and two velocities α', β'. A fancy way to see this as a 2-torus $\mathbb{T}^2$ where a 2-plane $\mathbb{R}^2$ is attached at every point. One calls this the tangent bundle.

25.2. The passage from three to four dimensions is important also because we only start to see in 4 dimensions the actual structure of differential calculus. Multivariable calculus courses like the one you just have seen, **conflate** and so **confuse** a lot of things which actually are different. For example, the curl of a vector field is a different object than a vector field and the divergence is is identified with the dual of the gradient. We already misrepresent things in single variable. Already the derivative f' in single variable calculus is actually a 1-form and not a function. The transition from dimension 3 to 4 clears up all confusions. There is little new conceptionally when going from 4 dimensions to 5 or higher dimensions as there is no need for a change of set-up any more.

25.3. To dwell on this, one can say that in many parts of topology or enumerative combinatorics, the three and four dimensional cases appear to be the most interesting. There are 5 platonic solids, regular discrete 2-dimensional spheres: tetrahedron, octahedron,hexahedron, dodecahedron, icosahedron. Then there are six regular discrete 3-dimensional Platonic solids 4 dimensions: are the 5-cell (hyper-tetrahedron), 8-cell (hyper-cube), 16-cell (hyper-octahedron), 24-cell, $120-cell$ (hyper-dodecahedron) and the 600-cell (hyper-icosahedron). In larger dimensions, only the cube, the tetrahedron or octahedron analogues exist.

25.4. When looking at the number $P(d)$ of $d-1$ dimensional Platonic solids in dimension d, one has $P(2)=\infty, P(3)=5, P(4)=6, P(5)=3, P(6)=3, P(7)=3, \dots$. These numbers indicate that "interest" peaks in dimension 4. The reason for the maximum is that there are two effects playing in: as higher dimensional the space is as more room we have and so more possibilities, but then there are also more constraints in the form of Dehn-Sommerville relations (curvature conditions) which constrain the possibilities. This is dramatic already when going from discrete 1-spheres to discrete 2-spheres, where the infinite set of polygons gets pushed to the 5 Platonic solids 5. This is Gauss-Bonnet. Now, because Gauss-Bonnet-Chern (the higher dimensional version) kicks in only when passing to even dimensional spheres, there are more 3-spheres (more opportunity) but then less 4-spheres (hyperspheres in $\mathbb{R}^4$ as more curvature conditions are added. It so happens that 3-spheres in $\mathbb{R}^4$ are the most interesting ones.

Geometric Objects

25.5. Geometric objects can be defined using **equations** as usual: we have the unit sphere $x^2+y^2+z^2+t^2=1$ for example, we have planes like $2x-2y+3z-5t=10$, cylinders $x^2+y^2+z^2=1$, paraboloids like $t=x^2+y^2+z^2$ or cones $t^2=x^2+y^2+z^2$. We have also two-dimensional surfaces like the 2-torus $t^2+x^2=1, y^2+z^2=1$ which is in this form is called the **Clifford torus**. Then there are 1-dimensional objects like this circle $x^2+y^2+z^2+t^2=3, x=1, y=1$. The 2-torus is an example of a product $\mathbb{T}\times\mathbb{T}=\mathbb{T}^2$, the cylinder is a product $\mathbb{T}\times\mathbb{R}$. The symmetric cylinder $\mathbb{T}^2\times\mathbb{R}^2$ is the phase space of the coupled pendulum.

25.6. We can also parametrize. Curves are parametrized as usual (we use u rather than t as a variable because t is our 4th coordinate)

$$\vec{r}(u)=[x(u),y(u),z(u),t(u)]\ .$$

The **3-sphere** can be parametrized with three angles as

$$\vec{r}(\phi,\theta,\psi)=[\cos(\psi)\cos(\theta),\cos(\psi)\sin(\theta),\sin(\psi)\cos(\theta),\sin(\psi)\sin(\theta)]$$

Hyper planes can be parametrized like with

$$\vec{r}(u,v,w)=[u+v,v-u,u+2w,u+v+w]\ .$$

The 2-surface $\vec{r}(\phi,\theta)=[\cos(\theta,\sin(\theta),\cos(\phi),\sin(\phi)]$ parametrizes the **Clifford torus**. A 3-torus $\mathbb{T}^3$ can be parametrized as

$$\vec{r}(\phi,\theta,\psi)=[(3+\cos(\psi))\cos(\theta),(3+\cos(\psi))\sin(\theta),(3+\sin(\psi))\cos(\theta),(3+\sin(\psi))\sin(\theta)]\ .$$

This is a parametrization close to the parametrization we are familiar with when embedding the torus in $\mathbb{R}^3$. The above parametrization is motivated by the Hopf parametrization of the sphere given above.

Fields

25.7. Real-valued functions f in four variables are also called **scalar functions** or 0-forms. Vector fields $\vec{F}(x,y,z,t)=[P,Q,R,S]$ are now also written as $\vec{F}=Pdx+Qdy+Rdz+Sdt$ and called 1-forms. There are now also 2-forms $Adxdy+Bdxdz+Cdtdx+Ddydz+Edtdy+Fdtdz$, 3-forms $Pdydzdt+Qdzdtdx+Rdtdxdy+Sdxdydz$ as well as 4-forms $gdxdydzdt$. There is a natural identification of 0-forms and 4-forms as well as 1-forms and 3-forms. To see how this fits in three dimensions, we would write the f for a 0-form, $F = Pdx + Qdy + Rdz$ for a 1-form $G = Pdydz + Qdzdx + Rdxdy$ for a 2-form and $gdxdydz$ for a 3-form. In 4-dimensions, it is the first time that we can no more just throw all k-forms into the to buckets **scalar field** or **vector field**. No more being able to confuse clears things up.

Derivatives

25.8. The **gradient** df of a function f is the 1-form $df = f_x dx + f_y dy + f_z dz + f_t dt$. The **curl** of a 1-form $F = Pdx + Qdy + Rdz + Sdt$ is defined as the 2-form $dPdx + dQdy + dRdz + dSdt = P_y dydx + P_z dzdx + P_t dtdx + Q_x dxdy + Q_z dzdy + Q_t dtdy + R_x dxdz + R_y dydz + R_t dtdz + S_x dxdt + S_y dydt + S_z dzdt = (Q_x - P_y)dxdy + (R_x - P_z)dxdz + (P_t - S_x)dtdx + (R_y - Q_z)dydz + (S_y - Q_t)dydt + (S_z - R_t)dzdt$. The **hyper-curl** of a 2-form $Adxdy + Bdxdz + Cdxdt + Ddydz + Edydt + Fdzdt$ is define as as $(D_t - E_z + F_y)dydzdt + (B_t - C_z + F_x)dzdtdx + (A_t - C_y + E_x)dtdxdy + (D_x - B_y + A_z)dxdydz$ The **divergence** of a 3-form $Pdydzdt + Qdzdtdx + Rdtdxdy + Sdxdydz$ is $(P_x + Q_y + R_z + S_t)dxdydzdt$.

25.9. You can now see the pattern. In general, the **exterior derivative** maps k-forms to $(k+1)$-forms. In four dimensions, there are 4 derivatives, the gradient, the curl, the hypercurl and the divergence. There seems to be no established word for the exterior derivative from 2 forms to 3-forms. We call it therefore the **hyper curl**, as it is some kind of dual or super curl. In general, if we write dI for any expression like dx, $dxdt$ or $dxdydt$, if $F = fdI$, then $dF = dfdI = (f_x dx + f_y dy + f_z dz + f_t dt)dI$, where $dxdI = d(xI) = (-1)^{|I|}d(Ix)$.

Integration

25.10. We have seen two kind of integrals appearing in two dimensions, the **line integral** $\int_a^b F(r(t)) \cdot r\ '(t)dt$ and the **double integral** $\iint_G f(u,v)\ dudv$. In three dimensions, we have besides the line integral also the **flux integral** $\iint_G F(r(u,v) \cdot r_u \times r_v\ dudv$ and the **triple integral** $\iiint_G f(u,v,w)\ dudvdw$. Let us express this with differential forms. For simplicity, we always just work with forms having one component. It is obvious how to extend this to sums by linearity. For a 1-form $F = Adx$, we integrate

$$\int_a^b A(r(u))x'(u)du\ .$$

For a 2-form $F = Adxdy$, we integrate $\iint_G A(r(u,v))\det(\begin{bmatrix} (r_u)_1 & (r_v)_1 \\ (r_u)_2 & (r_v)_2 \end{bmatrix}\ dudv$. For a 3-form $F = Adxdydz$, we integrate

$$\iiint_G A(r(u,v,w)\det(\begin{bmatrix} (r_u)_1 & (r_v)_1 & (r_w)_1 \\ (r_u)_2 & (r_v)_2 & (r_w)_2 \\ (r_u)_3 & (r_v)_3 & (r_w)_3 \end{bmatrix}\ dudvdw\ .$$

This determinant is the triple scalar product of the three column vectors. So, we can still work in principle without having seen any linear algebra course.

25.11. It should be clear how this is done for other components of the differential form. For $F = Bdy$ for example, we would compute the line integral as

$$\int_a^b B(r(u))y'(u)du \ .$$

For $F = Adxdzdt$ for example, we would compute the hyper flux as

$$\int\int\int_G A(r(u,v,w)\det(\begin{bmatrix} (r_u)_1 & (r_v)_1 & (r_w)_1 \\ (r_u)_3 & (r_v)_3 & (r_w)_3 \\ (r_u)_4 & (r_v)_4 & (r_w)_4 \end{bmatrix} \ dudvdw \ .$$

25.12. In general, a k-form F can be seen as as a rule which attaches to a point $\vec{x}$ a k-linear anti-symmetric map. So, $F(x)\cdot(v_1,\dots,v_k)$ is a number. And each of the maps $v_j \to F(x)\cdot(v_1,\dots,v_k)$ is linear and $F(x)\cdot(v_1,...,v_i,v_j,...,v_k) = -F(x)(v_1,...,v_j,v_i,...,v_k)$ for any pair i,j. Let $\vec{r}: G \to \vec{r}(G)$ be a parametrization, where G is a k-dimensional region mapped into $\mathbb{R}^4$. Now, we can assign to every point $\vec{x} = \vec{r}(\vec{u})$ the number $F(x)du = F(x)(\vec{r}_{u_1},\dots,\vec{r}_{u_k})$. The flux of the k-form through the k-surface $\vec{r}(G)$ is then defined as $\int_G F(r(\vec{u}))\cdot d\vec{u}$. For a 1-form F, we get the **line integral** $\int_a^b F(\vec{r}(t))\cdot\vec{r}'(t)\ dt$. For a 2-form F we get the **flux integral** $\int\int\int_G F(\vec{r}(u,v))\cdot(r_u,r_v)\ dudv$, for the 3-form F we get the **hyper flux integral** $\int\int\int_G F(\vec{r}(u,v,w))\cdot(r_u,r_v,r_w)\ dudvdw$. For the 4-form $Fdxdydzdt$, we get the **volume integral** $\int\int\int\int_G F(x,y,z,t)dxdydzdt$.

Stokes theorem

25.13. In four dimensions, we have four integral theorems. First of all, there is the **fundamental theorem of line integrals**

Theorem:
$$\int_a^b df(r(t))\cdot\vec{r}\,'(u)\ du = f(\vec{r}(b)) - f(\vec{r}(a))$$

25.14. Then there is **Stokes theorem** for a region G with boundary C

Theorem:
$$\int\int_G dF(r(u,v))\cdot(r_u,r_v)\ dudv = \int_C F(\vec{r}(u))\cdot\vec{r}\,'(u)\ du$$

25.15. New is the **hyper Stokes theorem**

Theorem:
$$\int\int\int_G dF(r(u,v,w)\cdot(r_u,r_v,r_w)\ dudvdw = \int\int_{\delta G} F(\vec{r}(u,v)\cdot(r_u,r_v)\ dudv$$

25.16. Finally, there is the **Divergence theorem**

Theorem:
$$\int\int\int\int_G dF(x,y,z,t)\ dxdydzdt = \int\int\int_{\delta G} F(\vec{r}(u,v,w))\cdot(r_u,r_v,r_w)\ dudvdw$$

25.17. All these integral theorems have the form

$$\int_G dF = \int_{\delta G} F ,$$

where F is a k-form and G is a $(k-1)$- dimensional surface. We see that now, when doing things in four dimensions, we have a frame work which easily generalizes to even higher dimensions. There are then just more variables and the determinants have to be evaluated with larger matrices. It is in 2 and 3 dimensions, where the situation had been **confusing**, the reason being that different spaces have been identified. The word "confusing" is spot-on because it means fusing things together which actually do need to be different. Still, pedagogically, it would still be difficult to get the general frame work running in a standard multi-variable course.

25.18. Historically, it took a while to get to this differential frame-work. It had been completely unknown in the 19th century even so, notions like "total differentials" $df = Adx + Bdy + Cdz$ were in wide use. The precise frame-work of differential forms was introduced in the modern form by Élie Cartan early in the 20'th century. Our presentation is more modern: we follow also Dieudonne who freed Stokes theorem from the concept of chains (which is a combinatorial formal way to define integration on manifolds). We use also the parametrizations as in multi-variable calculus and also avoid the usually initiated exterior algebra set-up which really only makes sense after having seen a serious linear algebra course. So, in principle, we could build a few modules on calculus in 4-dimension which directly follow the course, without the need of linear algebra.

25.19. We would like to continue here with some concrete computations similar to textbooks and hope to add this later. For now, let us just also mention that the frame work of differential forms is very elementary in the discrete, where one deals with graphs (V, E), a finite collection V of vertices connected with edges. Graphs are objects which kids already understand and are much easier to grasp than say planar geometry a la Euclid taught in elementary school. For a graph one can look at k-cliques, collections of nodes which are all connected to each other. If V is a set of people and E the set of (a, b) where a and b are friends, then a clique is a group of people where all are friends with other. A k-form F is now just a function F defined on the set of oriented k-cliques. The **boundary** δx of a clique x consists of all possible subcliques in which one person is taken away oriented in a compatible way with the main clique. The **dimension** of a clique is the number of members minus 1. A single person with no friends has dimension 0 for example. For a triangle $x = (a, b, c)$ (a clique of dimension 2), the boundary δx consists of three edges $(b, c), (c, a), (a, b)$ (each has dimension 1).

25.20. The **exterior derivative** $df(x)$ is defined as $f(\delta x)$ meaning that we sum up the values of f on all boundary parts. For example $df(a, b, c) = f(b, c) + f(c, a) + f(a, b)$. We see that the definition of exterior derivative is done in such a way that Stokes theorem $df(x) = f(\delta x)$ holds for a clique x with boundary δx. Now, a k-dimensional surface G is just a collection of k-dimensional cliques, where each is oriented. The **integal** $\int_G F$ plays the role of the k-dimensional flux. Stokes theorem $\int_G dF = \int_{\delta G} F$ is now essentially a tautology. The space of k-forms forms a f_k-dimensional space if f_k is the number of k dimensional cliques in the graph. The exterior derivatives can now be implemented as concrete matrices d_k. One can more conveniently use all forms together and write one large $f_0 + f_1 + ... + f_d$ matrix d as the exterior derivative. The matrix $D = d + d^*$ is then the **Dirac operator** and $L = D^2 = d^*d + dd^*$ is the **Laplacian**.

25.21. Now one can go on and do physics. Solutions of the discrete PDE $Lu = 0$ (the Laplace equation) are called **harmonic forms**. One can look at the **heat equation** $u_t = -Lu$ or the **Schrödinger equation** $i\hbar u = -Lu$ or the **wave equation** $u_{tt} = -Lu$ and write this as $(\partial_t - iD)(\partial_t + iD)u = 0$ showing that a solution of the wave equation is a super position of two **Dirac equations** $iu_t = Du$ and $iu_t = -Du$, which are Schrödinger type equations. The Maxwell equations $F = dA, d^*F = j$ can now be considered and a graph. Like the actual Maxwell equations, one gets so in a Lorentz gauge $d^*A = 0$ to the Laplace type equation $LA = 0$. If the graph is seen as space-time, then we have defined **electro magnetic waves** on the graph. We have switched on light on the graph.

25.22. We see that large chunks of rather serious physics becomes simple. One can imagine that other civilisations do not bother with continuum calculus and directly do all physics with graphs, doing combinatorially what we do anyway eventually when we implement stuff on computers (which are finite entities). This brings us to Egan's novel Schilds ladder which is serious hard core science ficton imagining such civilisations. There are also here on earth some scientists who pursue physics on finite spaces. An example is loop quantum gravity.

Oliver Knill, knill@math.harvard.edu, Math S-21a, Harvard Summer School, 2020