MULTIVARIABLE CALCULUS

MATH S-21A

Unit 1: Geometry and Distance

Lecture

1.1. A point on the **real line** $\mathbb{R}$ is given by a single coordinate x. If x is positive, it is located on the **positive axis** which is divided by zero 0 from the **negative axis**, places where the coordinates are negative. A point P in the **plane** $\mathbb{R}^2$ has two **coordinates** and is written as $P = (x, y)$. A point in space $\mathbb{R}^3$ is determined by three coordinates and written as $P = (x, y, z)$. The signs of the coordinates define four **quadrants** in $\mathbb{R}^2$ or eight **octants** in $\mathbb{R}^3$. These regions all intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are bound by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

1.2. In $\mathbb{R}^2$ it is custom to orient the x-axis to the "east" and the y-axis to the "north". In $\mathbb{R}^3$, the most common coordinate system is to see the xy-plane as the "ground" and imagine the z-coordinate axes pointing "up". In computer graphics or photography, the xy-plane represents the **retina** or film plate and the z-coordinate measures the distance towards the viewer. In this **photographic coordinate** system, your eyes and chin define the plane $z = 0$ and the nose points in the positive z direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0)$ for the right eye, $(-1, 0, 0)$ for the left eye, then the tip of your nose might have the coordinates $(0, -1, 1)$.

1.3. The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as

Definition: $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

Note that this is a **definition** and not a result. It is motivated by the **theorem of Pythagoras**, but we will **prove** the later result in a moment. This distance is defined in any dimension. In the plane for example the distance of the point (x, y) to (a, b) is $\sqrt{(x-a)^2 + (y-b)^2}$. If we work in $\mathbb{R}^2$, we do not think of it as part of $\mathbb{R}^3$. Coordinates work in arbitrary dimensions. A collection of n data points defines a vector in $\mathbb{R}^n$. Working in Euclidean space $\mathbb{R}^n$ makes sense from a **data scientist point of view**. One can define the Euclidean distance between $x = (x_1, \dots, x_n)$ and $a = (a_1, \dots, a_n)$ as $d(x, a)^2 = \sum_{k=1}^n (x_i - a_i)^2$. Having the sum of the squares appears in statistics in **least square problems**.

1.4. **Points**, **curves**, **surfaces** and **solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a **line**, an example of a surface is a **plane**, an example of a solid is the **ball**, the interior of a **sphere**.

Definition: A **circle** of radius $r \geq 0$ centered at $P = (a, b)$ is the collection of points in $\mathbb{R}^2$ which have distance r from P. A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in $\mathbb{R}^3$ which have distance $\rho \geq 0$ from P. The equation of a sphere is $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

1.5. When **completing the square** of an equation $x^2 + bx + c = 0$, we add $(b/2)^2 - c$ on both sides of the equation in order to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$. This is the **quadratic equation**. Know this equation. You don't want to waste your creative power having to re-derive this again and again.

Examples

1.6. $P = (-2, -3)$ is in the third quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -8)$ is located on the negative z axis. The point $P = (1, 2, -3)$ is below the xy-plane. Can you spot the point Q on the xy-plane which is closest to P?

1.7. Problem: Find the distance midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

1.8. The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after a **completion of the square** $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.

1.9. The distance $d(P, Q) = |x - a| + |y - b|$ in the plane $\mathbb{R}^2$ is called the **taxi metric** or **Manhattan distance**. **Problem**: draw a circle of radius 2. More challenging is to draw an ellipse: the set of points whose sum of the distances from $(-2, 0)$ and $(2, 0)$ is equal to 6. You can do that with a neat geometric construction.

1.10. Draw the unit circle of the **quartic distance** $d(x, y) = (x - a)^4 + (y - b)^4$. More generally, for any $p > 1$, we get a distance $d(x, y) = (x - a)^p + (y - b)^p$. For $p = 1$, it is the **taxi metric**, for $t = 2$ it is the **Euclidean metric**, for $t \to \infty$ it goes to the distance $\max(|x - a|, |y - b|)$ which is the l^∞ metric. **Problem**: is $d(P, Q) = \sqrt{|x - a|} + \sqrt{|y - b|}$ a distance? **Answer**: no, while it satisfies $d(P, Q) = d(Q, P)$ and is zero if and only if $P = Q$, it does not satisfy the **triangle inequality** $d(A, B) + d(B, C) \geq d(A, C)$. We call a space (X, d) for which d is a distance formula satisfying $d(P, Q) = d(Q, P)$, $d(P, Q) = 0 \leftrightarrow P = Q$ and $d(A, B) + d(B, C) \geq d(A, C)$ a **metric space**.

1.11. Problem: Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1, 0)$ and $B = (-1, 0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a, 0)$ satisfies this equation, $d(P, A) + d(P, B) = (a - 1) + (a + 1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0, b)$ satisfying it gives $d(Q, A) + d(P, B) = 2\sqrt{b^2 + 1} = 3$ or $b = \sqrt{5}/2$.

1.12. In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. But ideas do not grow in a vacuum; Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. One of the first to explore higher dimensional Euclidean space was Ludwig Schläfli. [1]

1.13. The method of completion of squares is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [2]

1.14. The Euclidean geometry described is only one of many geometries. One can work with more general **metric spaces**. An important class of metric spaces are studied in Riemannian geometry, where the distance between two points can become dependent on where we are. Space becomes curved. This is the frame work of general relativity. Formally, this can happen by changing the coefficients E, G of the metric $d(P, Q)^2 = E(x - a)^2 + G(y - b)^2$. One a sphere, where $x = \theta \in [0, 2\pi]$ is longitude and $y = \phi \in [0, \pi]$ is latitude, one would take $E = \sin^2(y), G = 1$. Two points on the arctic circle with fixed longitude have shorter distance than two points on the equator with the same fixed longitudes. It is important to think now of the surface of the sphere as a space itself, without its embedding in the ambient space. This space is curved. Our four dimensional space-time universe is curved depending on the matter distribution.

Homework

This homework is due on Tuesday, 7/2/2019.

Problem 1.1: Describe in words and draw the objects in $\mathbb{R}^3$.
a) $(z - 1)^2 + (x - 2)^2 = 1$. b) $|x - 1| + |y - 2| + |z - 3| = 4$.
c) $x^2 y^2 z^2 = 0$, d) $x + 2y + 3z = 6$.
e) $|(x - 1, y, z)| - |(x, y - 1, z)| = 1$. f) $x^2 - z^2 = 1$.

[1] An entertaining read is "Descartes secret notebook" by Amir Aczel which deals with an other discovery of Descartes.

[2] The book "The mathematics of Egypt, Mesopotamia, China, India and Islam, by Ed Victor Katz, page 542 contains translations of some of this work.

Problem 1.2: A data point $P = (5, 6, 7)$ gives temperature, rain and wind velocity. We can visualize this in the (t, r, v) space. a) Find the distance of P to the t-axes, where r, v are zero.
b) Find the center of the sphere $t^2 + r^2 - 6r + v^2 = 10$.

Problem 1.3: Verify that the radius of the inscribed circle in a $3 : 4 : 5$ triangle is 1. Here is a possible hint: make a picture of the triangle ABC given by $A = (0, 0), B = (4, 0), C = (0, 3)$, introduce $M = (1, 1)$ then get the coordinates of the points X, Y, Z then compute the distances to verify that the inscribed circle touches the triangle at X, Y, Z.

Problem 1.4: The figure shows rectangles of area 64 and 65 made up of matching pieces. What is going on? It is a famous and classical problem. Try first on your own!

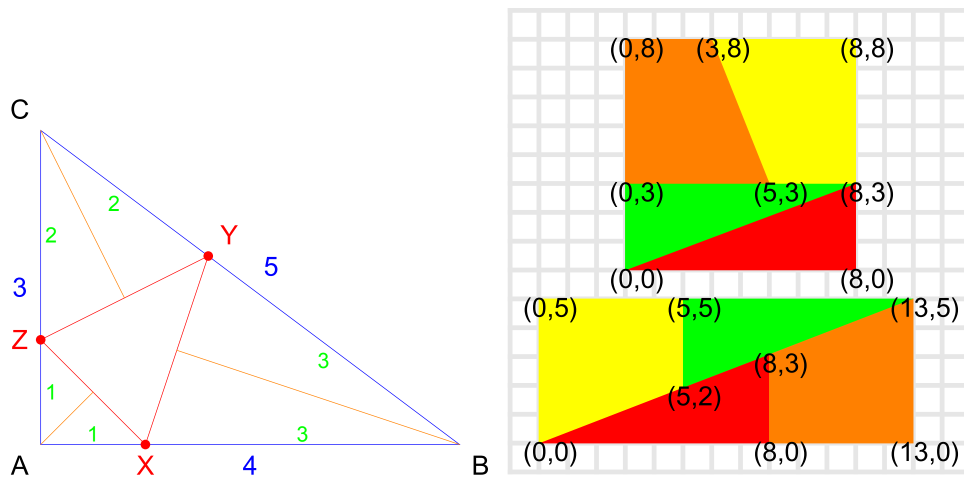

FIGURE 1. The 3-4-5 triangle and the missing square riddle.

Problem 1.5: You play billiard in the table $\{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 8\}$. a) Hit the ball at $(3, 2)$ to reach the hole $(4, 8)$ bouncing 3 times at the left wall and three times at the right wall and no other walls. Find the length of the shot.
b) Hit from $(3, 2)$ to reach the hole $(4, 0)$ after hitting twice the left and twice the right wall as well as the top wall $y = 8$ once. What is the length of the trajectory?

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Unit 2: Vectors and dot product

LECTURE

2.1. Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix}$.
We simply write also $[x-a, y-b, z-c]^T$ or $\langle x-a, y-b, z-c\rangle$ in order to save space. It is custom in linear algebra to call $[3, 4, 5]$ a row vector and its transpose $[3, 4, 5]^T = \langle 3, 4, 5\rangle$ a **column vector**. As the vector starts at P to Q we write $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in $\vec{v} = [p, q, r]^T$ are called the **components** of $\vec{v}$.

2.2. Vectors can be placed **anywhere** in space. [1] Two vectors with the same components are considered **equal**. Vectors can be translated into each other if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = [p, q, r]^T$ heads to the point (p, q, r). One can therefore identify **points** $P = (a, b, c)$ with **vectors** $\vec{v} = [a, b, c]^T$ attached to the origin. For clarity, we often draw an arrow $\vec{}$ on top of a vector variable and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to write $[2, 3, 4]^T$ or $\langle 2, 3, 4\rangle$ for vectors and $(2, 3, 4)$ for points.

2.3.

Definition: The **sum** of two vectors is $\vec{u} + \vec{v} = [u_1, u_2]^T + [v_1, v_2]^T = [u_1 + v_1, u_2 + v_2]^T$. The **scalar multiple** is $\lambda\vec{u} = \lambda[u_1, u_2]^T = [\lambda u_1, \lambda u_2]^T$. The **difference** $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1)\cdot\vec{v}$.

Commutativity, associativity, or distributivity rules for vectors are inherited directly from the corresponding rules for numbers.

2.4. The vectors $\vec{i} = [1, 0, 0]^T$, $\vec{j} = [0, 1, 0]^T$, $\vec{k} = [0, 0, 1]^T$ are called **standard basis vectors**. We will avoid this notation mostly but it has historically grown as some notions like the dot and cross product have grown from **quaternions** which are points (t, x, y, z) in $\mathbb{R}^4$ usually written as $t + ix + jy + kz$.

[1] In differential geometry a vector $\vec{v}$ is seen in the so called tangent space to a point P.

2.5.

Definition: The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P,Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is called a **direction** of $\vec{v}$. The only vector of length 0 is the 0 vector $[0,0,0]^T$.

2.6.

Definition: The **dot product** of two vectors $\vec{v} = [a,b,c]^T$ and $\vec{w} = [p,q,r]^T$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

2.7. Different notations for the dot product are used in different mathematical fields. While mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, the **Dirac notation** $\langle \vec{v} | \vec{w} \rangle$ is used in quantum mechanics or the **Einstein notation** $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity is used. In statistics, it is called the **covariance** $\mathrm{Cov}[v,w]$ of centered data points. The dot product is also called **scalar product** or **inner product**. It could be generalized. Any product $g(v,w)$ which is linear in v and w and satisfies the symmetry $g(v,w) = g(w,v)$ and $g(v,v) \geq 0$ and $g(v,v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v,w) = 2v_1w_1 + 3v_2w_2 + 5v_3w_3$.

2.8. The dot product determines distances and distances determines the dot product.
Proof: Write $v = \vec{v}$. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v+w)\cdot(v+w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

$$v \cdot w = (|v+w|^2 - |v|^2 - |w|^2)/2 \; .$$

2.9. The **Cauchy-Schwarz inequality** is

Theorem: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. If $|w| = 0$, the statement holds as both sides are zero. Otherwise, assume $|w| = 1$ by dividing the equation by $|w|$. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw)\cdot(v - aw)$ to get $0 \leq (v-(v\cdot w)w)\cdot(v-(v\cdot w)w) = |v|^2 + (v\cdot w)^2 - 2(v\cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$. □

2.10. Having established this, it is possible to give a definition of what an **angle** is, without referring to any geometric pictures:

Definition: The **angle** between two nonzero vectors $\vec{v}, \vec{w}$ is defined as the unique $\alpha \in [0,\pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$. Since cos maps $[0,\pi]$ in a 1:1 manner to $[-1,1]$, this is well defined.

2.11. The **Al Kashi's theorem** gives the third side length c of a triangle ABC in terms of the sides $a = d(B,C), b = d(A,C)$ and α, the angle at the vertex C

Theorem: $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$. □

2.12. The **triangle inequality** tells

Theorem: $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof. $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$. □

Definition: Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]^T$ is orthogonal to $\vec{w} = [-3, 2]^T$.

2.13. We can now prove the **Pythagoras theorem**:

Theorem: If $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof. $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. □

2.14.

Definition: The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.

2.15. The projection allows to **visualize** the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Examples

2.16. For example, with $\vec{v} = [0, -1, 1]^T$, $\vec{w} = [1, -1, 0]^T$, $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]^T$. Its length is $1/\sqrt{2}$.

2.17. The RGB **color space** consists of triples $\vec{v} = [r, g, b]^T$ describing the amount of red, green and blue of a **color**. An other coordinate system is the CMY **color space** consisting of triples $\vec{v} = [c, m, y]^T = [1-r, 1-g, 1-b]^T$, where c is **cyan**, m is **magenta** and y is **yellow**.

2.18. In physics, forces and fields $\vec{F}$ are described by vectors. The **velocity** of a curve $r(t) = [x(t), y(t), z(t)]$ is a vector attached to the point $r(t)$.

2.19. In probability theory, data are described by vectors. One calls them also **random variables**. It is in statistics, where higher dimensional spaces appear.

Homework

This homework is due on Tuesday, 7/2/2019.

Problem 2.1: Find a **unit vector** parallel to $\vec{u} + 2\vec{v} + 4\vec{w}$ if $\vec{u} = [-10, 2, 9]^T$ and $\vec{v} = [1, 1, 3]^T$ and $\vec{w} = [3, 1, 1]^T$.

Problem 2.2: An **Euler brick** is a **cuboid** with side lengths a, b, c such that all face diagonals are integers.
a) Verify that $\vec{v} = [a, b, c]^T = [275, 252, 240]^T$ is a vector which leads to an Euler brick. Halcke found the first one in 1719.
b) (*) Verify that $[a, b, c]^T = [u(4v^2 - w^2), v(4u^2 - w^2), 4uvw]^T$ leads to an Euler brick if $u^2 + v^2 = w^2$.
(Sounderson 1740) If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

Problem 2.3: **Colors** are encoded by vectors $\vec{v} = [$ red , green , blue $]^T$. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.
a) Determine the angle between the colors yellow and magenta.
b) What is the vector projection of the magenta-orange mixture $\vec{x} = (\vec{v} + \vec{w})/2$ onto green $\vec{y}$?

Problem 2.4: A rope is wound exactly 5 times around a stick of circumference 1 and length 12. How long is the rope?

Problem 2.5: a) Find the angle between the main diagonal of the unit cube and one of the face diagonals. Assume that both diagonals pass through a common vertex.
b) Find the vector projection of the main diagonal $\vec{v} = [1, 1, 1]^T$ onto the side diagonal $\vec{w} = [1, 1, 0]^T$.
c) Find the scalar projection of $\vec{v}$ on $\vec{w}$.

Postscript: Coordinates and Data

2.20. We live in a time, where **data** are increasingly important. This is good news for multi-variable calculus, as data points are usually given as points in an Euclidean space and analyzed using tools of multi-variable calculus. There are various ways how data can be stored, in a computer as lists of **bits**, in a quantum computer as a list of **qbits**, in a **relational database** as a list of tables, in a **graph data base** as a list of graphs. A sort of graph database has been designed already by the Incas in the form of **Khipu**, which are also called "talking knots". In a picture, data are color values attached in an array, a **song** is an array of amplitudes, a movie is an array of pictures, which each is an array of color vectors. It does not matter, in the end, we can store information as a point in a Euclidean space. The Harvard Khipu data base for example is a standard relational database encoding the three dimensional knots which are still available. We look here at an example, which illustrates the **story of data**:

2.21. The next table shows some data from the 2018 rankings of the top universities. The data were obtained from the website

$$https://www.topuniversities.com/university-rankings/world-university-rankings/2018$$

There are different such rankings. It is important that you are aware about how arbitrary such a "ranking" can be done. This is especially if you think about how the data were actually obtained. Which of the data can be considered objectively reproducible, which ones are more subjective?

2.22. The following concrete data provide for each institution a vector:

[Overall,Academic,Employer,Faculty/Student ratio,Citations,Int. faculty, Int. students]

MIT	100	100	100	100	99.8	100	95.5
Stanford	98.6	100	100	100	99	99.8	70.5
Harvard	98.5	100	100	99.3	99.8	92.1	75.7
Caltech	97.2	98.7	81.2	100	100	96.8	90.3
Oxford	96.8	100	100	100	83	99.6	98.8
Cambridge	95.6	100	100	100	77.2	99.4	97.9
ETH	95.3	98.2	96.2	82.4	98.7	100	98.6
Imperial	93.3	98.7	99.9	99.9	67.8	100	100
Chicago	93.2	99.6	90.7	97.4	83.6	74.2	82.5
UCL	92.9	99.3	99.2	99.2	66.2	98.7	100

2.23. Can you figure out how the first entry is computed? It is an average of the 6 entries
$\vec{x} = [Academic, Faculty - StudentRatio, Citations, International, IntStudents]$.
There is a mystery vector $\vec{v} = [a, b, c, d, e, f]$ which averages the ranking as a **dot product** of $\vec{x}$ with $\vec{v}$. Check also

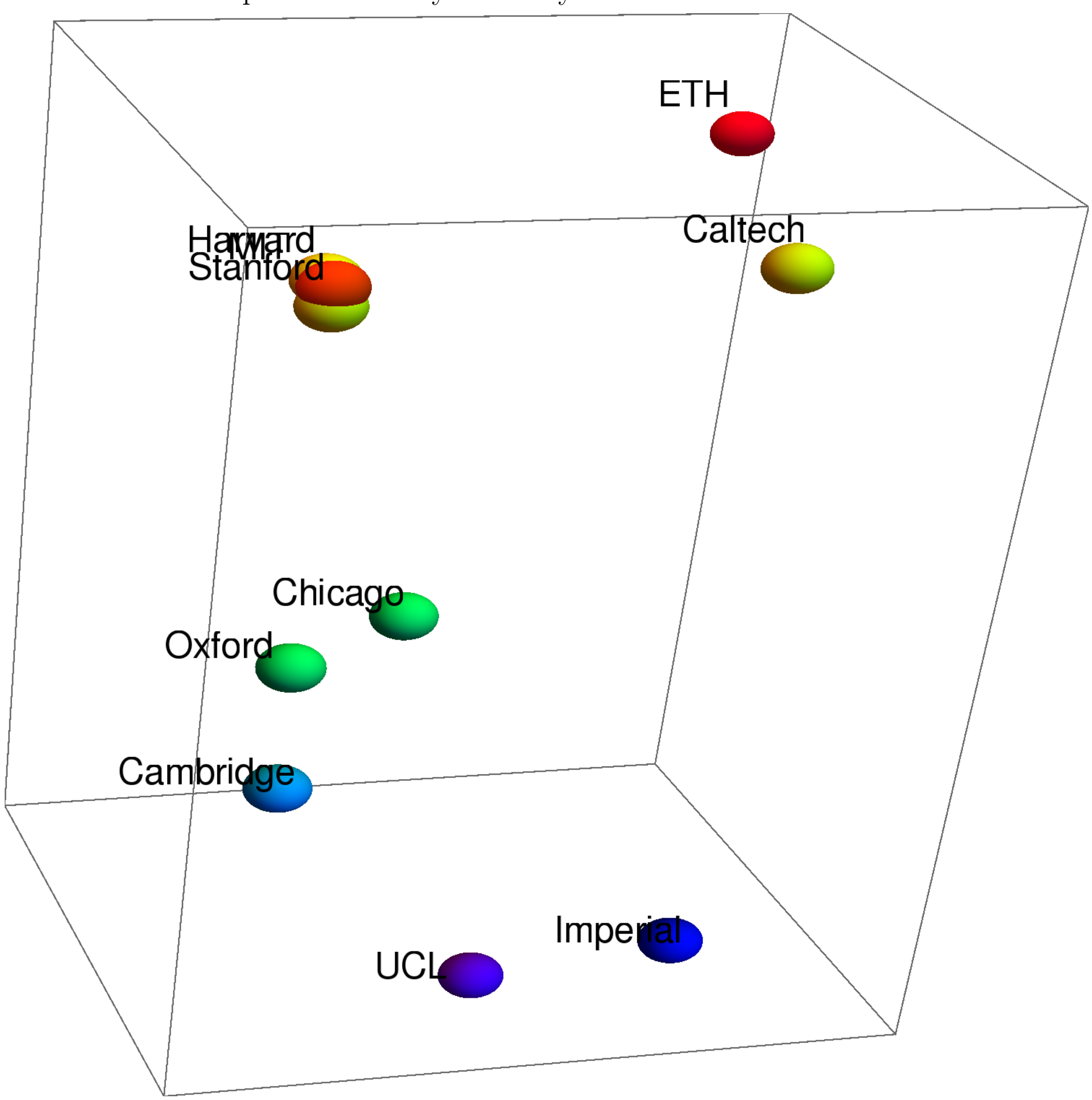
$$https://www.topuniversities.com/qs-world-university-rankings/methodology.$$

2.24. Obviously, the overall ranking depends on the key. There is a ranking key so that the university UCL is on the top for example. Can you find one? This requires to find a ranking vector $\vec{v}$ for which $\vec{x} \cdot \vec{v}$ is maximal for UCL.

2.25. How can we **visualize data**? In the next figure, we plot three relevant data points given by

$$\vec{x} = [Academic, Faculty - StudentRatio, Citations]$$

By comparing the data with the plot, can you figure out, what each of the coordinate axes is? Also this visual representation produces some kind of ranking. Being a graduate of ETH myself and being at Harvard, having been at Caltech, the picture has been turned so that these universities look particularly good ... It is important that you are aware of such manipulations. They are everywhere!



The mystery vector is $\vec{v} = [0.4, 0.1, 0.2, 0.2, 0.05, 0.05]$.

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Unit 3: Cross product

LECTURE

3.1. The **cross product** of two vectors $\vec{v} = [v_1, v_2]^T$ and $\vec{w} = [w_1, w_2]^T$ in the plane is the **scalar** $\vec{v} \times \vec{w} = v_1 w_2 - v_2 w_1$. To remember this, you can write it as a determinant of a 2×2 matrix $A = \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$, which is the product of the diagonal entries minus the product of the side diagonal entries.

3.2.

Definition: The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]^T$ and $\vec{w} = [w_1, w_2, w_3]^T$ in space is defined as the **vector**

$$\vec{v} \times \vec{w} = [v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1]^T .$$

To remember this, we can write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$ using the notation $\vec{i} = [1, 0, 0]$, $\vec{j} = [0, 1, 0]$ and $\vec{k} = [0, 0, 1]$.

3.3. Examples: the cross product of $[1, 2]^T$ and $[4, 5]^T$ is $5 - 8 = -3$. The cross product of $[1, 2, 3]^T$ and $[4, 5, 1]^T$ is $[-13, 11, -3]^T$. The cross product is clearly anti-commutative: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

Theorem: In $\mathbb{R}^3$, the vector $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$ and has length $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof. To see the orthogonality, verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$. To check the length formula, we use the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ which is also called **Cauchy-Binet** formula. We will do that by direct computation in class. To finish up, use $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$. ☐

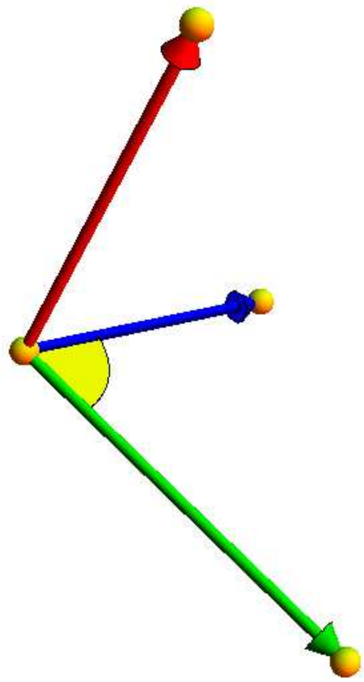

FIGURE 1. The cross product produces a vector perpendicular to two vectors. The length of the vector is the area of the parallelogram.

3.4. The statement can intuitively also to be seen by choosing a coordinate system in which the vectors are given as In that special case $\vec{v} = [a, 0, 0]^T$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]^T$, we have $\vec{v} \times \vec{w} = [0, 0, ab\sin(\alpha)]^T$ which has length $|ab\sin(\alpha)|$. This argument however assumes that the cross product does not change, if we change the coordinate system.

3.5. The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. As stated as a **definition**, nothing needs to be proven. The definition fits with our common intuition we have about area because $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$.

3.6. The **trigonometric sin-formula** relates the side lengths a, b, c and angles α, β, γ of a general triangle:

Theorem: $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.

Proof. We can express the area of the triangle in three different ways:

$$ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta) .$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity. □

3.7. It follows from the sin-formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$ that $\vec{v} \times \vec{w}$ is zero if and only if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda \vec{w}$ for some real λ. The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are considered parallel even so sometimes the notion **anti-parallel** is used.

3.8.

Definition: The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** of three vectors is defined as the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors define a **right-handed** coordinate system.

3.9. Again, there was no need to prove anything because we **defined** volume and orientation. Why does this fit with our intuition? The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system. Since the triple scalar product is linear with respect to each vector, we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with twice the volume.

Examples

3.10. Problem: Find the volume of the parallelepiped which has the vertices $O = (1,1,0), P = (2,3,1), Q = (4,3,1), R = (1,4,1)$. **Answer**: We first see that the solid is spanned by the vectors $\vec{u} = [1,2,1]^T$, $\vec{v} = [3,2,1]^T$, and $\vec{w} = [0,3,1]^T$. We get $\vec{v} \times \vec{w} = [-1,-3,9]^T$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

3.11. Problem: Two apples have the same shape, but one has a 3 times larger diameter. What is their weight ratio? **Answer**. For a parallelepiped spanned by $[a,0,0]^T$ $[0,b,0]^T$ and $[0,0,c]^T$, the volume is the triple scalar product abc. If a, b, c are all tripled, the volume gets multiplied by a factor 27. Now cut each apple into the same amount of parallelepipeds, the larger one with slices 3 times as large too. Since each of the pieces has 27 times the volume, also the apple is 27 times heavier!

3.12. Problem. A **3D scanner** is used to build a 3D model of a face. It detects a triangle which has its vertices at $P = (0,1,1), Q = (1,1,0)$ and $R = (1,2,3)$. Find the area of the triangle. **Solution.** We have to find the length of the cross product of $\vec{PQ}$ and $\vec{PR}$ which is $[1,-3,1]^T$. The length is $\sqrt{11}$.

3.13. Problem. The scanner now detects an other point $A = (1, 1, 1)$. On which side of the triangle is it located if the cross product of $\vec{PQ}$ and $\vec{PR}$ is considered the direction "up". **Solution.** The cross product is $\vec{n} = [1, -3, 1]^T$. We have to see whether the vector $\vec{PA} = [1, 0, 0]^T$ points into the direction of $\vec{n}$ or not. To see that, we have to form the dot product. It is 1 so that indeed, A is "above" the triangle. Note that a triangle in space a priori does not have an orientation. We have to tell, what direction is "up". That is the reason that file formats for 3D printing like contain the data for three points in space as well as a vector, telling the direction.

Homework

This homework is due on Tuesday, 7/2/2019.

Problem 3.1: a) Find a unit vector perpendicular to the space diagonal $[1, 1, 1]^T$ and the face diagonal $[1, 1, 0]^T$ of the cube.
b) Find the volume of the parallelepiped for which the base parallelogram is given by the points $P = (5, 2, 2), Q = (3, 1, 2), R = (1, 4, 2), S = (-1, 3, 2)$ and which has an edge connecting P with $T = (5, 6, 8)$.
c) Find the area of the base and use b) to get the height of the parallelepiped.

Problem 3.2: a) Assume $\vec{u}+\vec{v}+\vec{w} = \vec{0}$. Verify that $\vec{u}\times\vec{v} = \vec{v}\times\vec{w} = \vec{w}\times\vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.

Problem 3.3: To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $[a, b, c]^T$ normal to the plane and fix d so that P is in the plane.

Problem 3.4: Verify the "BAC minus CAB" formula (due to Lagrange) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space.

Problem 3.5: A product $*$ is said to satisfy the **cancellation property** if for all $x, y, z \neq 0$: $x * z = y * z$ implies that $x = y$.
a) Does the dot product satisfy the cancellation property?
b) Does the cross product satisfy the cancellation property?

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MULTIVARIABLE CALCULUS

MATH S-21A

Unit 4: Lines and Planes

Lecture

4.1. A point $P = (p, q, r)$ and a vector $\vec{v} = [a, b, c]^T$ define the **line**

$$L = \{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ t \in \mathbb{R} \} .$$

The line consists of all points obtained by adding a multiple of the vector $\vec{v} = [a, b, c]^T$ to the vector $\vec{OP} = [p, q, r]^T$. It contains the point P as well as a copy of $\vec{v} = \vec{PQ}$ attached to P. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". At $t = 0$, we are at the end point P of $\vec{OP}$ and at $t = 1$, we are at the end point Q of $\vec{OQ} = \vec{OP} + \vec{v}$.

4.2. If t is restricted to values in a **parameter interval** $[t_1, t_2]$, then $L = \{[p, q, r]^T + t[a, b, c]^T, \ t_1 \leq t \leq t_2 \ \}$ is a **line segment** which connects $\vec{r}(t_1)$ with $\vec{r}(t_2)$. For example, to get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, form the vector $\vec{v} = \vec{PQ} = [1, 3, 4]^T$ and get $L = \{[x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T ; \}$. This can be written also as $\vec{r}(t) = [1+t, 1+3t, 2+4t]^T$. If we write $[x, y, z]^T = [1, 1, 2]^T + t[1, 3, 4]^T$ as a collection of equations $x = 1+2t, y = 1+3t, z = 2+4t$ and solve the first equation for t:

$$L = \{(x, y, z) \mid (x-1)/2 = (y-1)/3 = (z-2)/4 \ \} .$$

4.3. The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]^T$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x-p}{a} = \frac{y-q}{b} = \frac{z-r}{c} .$$

The reason is that each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$. The interpretation is that the line is written as an intersection of two planes.

4.4. A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.
An example is $\Sigma = \{[x, y, z]^T = [1, 1, 2]^T + t[2, 4, 6]^T + s[1, 0, -1]^T \ \}$. This is called the **parametric description** of a plane.

4.5. If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = [a, b, c]^T$, this means $ax + by + cz = ax_0 + by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have shown:

Theorem: The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point P is $ax + by + cz = d$, where $[a, b, c]^T = \vec{v} \times \vec{w}$ and where d is obtained by plugging in P.

4.6. Problem: Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = \vec{PQ} = [1, 2, 0]^T$ and $\vec{w} = \vec{PR} = [2, 2, 2]^T$. The normal vector $\vec{n} = \vec{v} \times \vec{w} = [4, -2, -2]^T$ leads to the equation $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of one of the points. In our case, it is $4x - 2y - 2z = -4$.

4.7. Problem: Find the angle between the planes $x + y = -1$ and $x + y + z = 2$, The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two normal vectors $\vec{n} = [a, b, c]^T$ and $\vec{m} = [e, f, g]^T$.
Answer: find the angle between $\vec{n} = [1, 1, 0]^T$ and $\vec{m} = [1, 1, 1]^T$. It is $\arccos(2/\sqrt{6})$.

Examples

4.8. To practice the concepts, we look at **distance formulas**.

1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|[-7, 0, -3]^T \cdot [2, 4, 5]^T|}{|[2, 4, 5]^T|} = \frac{29}{\sqrt{45}} .$$

2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2, 3, 1)$ to the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$, compute

$$d(P,L) = \frac{|[-1,-2,1]^T \times [5,0,1]^T|}{[5,0,1]^T} = \frac{|[-2,6,10]^T|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}} .$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$ form the cross product of $[-1,1,0]^T$ and $[5,1,2]^T$ is $[2,2,-6]^T$. The distance between these two lines is

$$d(L,M) = \frac{|(3,1,2) \cdot (2,2,-6)|}{|[2,2,-6]^T|} = \frac{4}{\sqrt{44}} .$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma, \Pi) = \frac{|e-d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

$$\frac{|8-1|}{|[5,4,3]^T|} = \frac{7}{\sqrt{50}} .$$

FIGURE 1. The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites.

Homework

This homework is due on Tuesday, 7/2/2019.

Problem 4.1: Given the three points $P = (7, 4, 5)$ and $Q = (1, 3, 9)$ and $R = (4, 2, 10)$. find the parametric and symmetric equation for the line perpendicular to the triangle PQR passing through its center of mass $(P + Q + R)/3 = (4, 3, 8)$.

Problem 4.2: A regular tetrahedron has vertices at the points $P_1 = (0, 0, 6)$,$P_2 = (0, \sqrt{32}, -2)$, $P_3 = (-\sqrt{24}, -\sqrt{8}, -2)$ and $P_4 = (\sqrt{24}, -\sqrt{8}, -2)$. Find the distance between two edges which do not intersect.

Problem 4.3: Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1+4t, y = 1-4t, z = 8t$ and intersects this line in a point Q.

Problem 4.4: Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.

Problem 4.5: a) Find the distance between the point $P = (3, 3, 4)$ and the line $2x = 2y = 2z$.
b) Parametrize the line $\vec{r}(t) = [x(t), y(t), z(t)]^T$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify that the minimal value agrees with a).

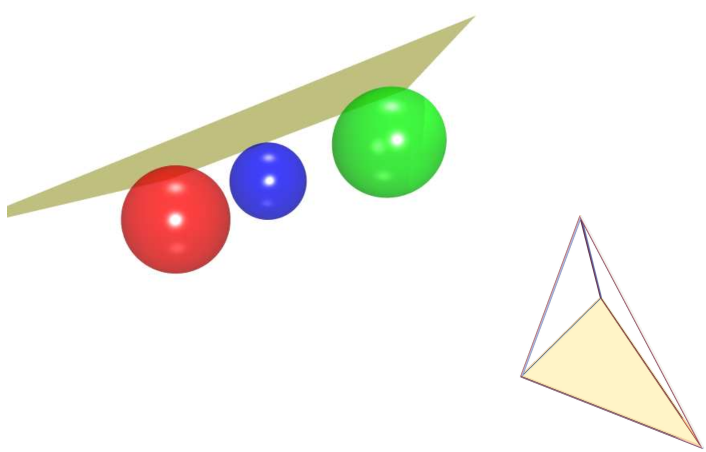

Figure 2. The sphere problem and the the tetrahedron.

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