7/12/2018 FIRST HOURLY Practice 9 **Maths 21a, O.Knill, Summer 2018**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** If two planes $ax + by + cz = d$ and $Ax + By + Cz = D$ are parallel then $a = A, b = B$, and $c = C$.

Solution:

We can have $A = 2a, B = 2b, C = 2c, D = 2d$ for example.

2) **T** F The point $(x, y, z) = (1, 1, \sqrt{2})$ has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$.

Solution:

Use the transformation formula.

3) **T** F Every point on the parametric curve $\vec{r}(t) = [t, t^2, -t]^T$ lies on the surface $xz + y = 0$.

Solution:

Check with $x = t, y = t^2, z = -t$.

4) **T** F The two surfaces $f(x, y, z) = 3$ and $f(x, y, z) = 5$ of the function $f(x, y, z) = 2x^2 + y^3 + z^4$ do not intersect at any point in space.

Solution:

The function is smooth so that level surfaces to different values can not intersect.

5) T **F** $\vec{u} \times \vec{i}$ and $\vec{u} \times \vec{j}$ are perpendicular for all vectors $\vec{u}$.

Solution:

Take $\vec{u} = \vec{i} + \vec{j}$.

6) T **F** If $\vec{u}$ and $\vec{v}$ are parallel then $\vec{u} \cdot \vec{v} \geq |\vec{u} \times \vec{v}|$.

Solution:

We can have $\vec{u} = -\vec{v}$ in which case the left hand side is negative if $\vec{v}$ has positive length.

7) T **F** If a surface has the property that all intersections with the planes $y = constant$ are straight lines, then the surface is a plane.

Solution:

Take the function $y = x^2$ for example. Its graph is not a plane but $y = constant$ are lines.

8) T **F** For any non-zero vectors $\vec{u}$ and $\vec{w}$, we must have $\text{proj}_{\vec{u}}\vec{w} = -\text{proj}_{\vec{w}}\vec{u}$.

Solution:

The projection onto $\vec{u}$ is parallel to $\vec{u}$ and the projection onto $\vec{v}$ is parallel to $\vec{v}$.

9) T **F** In the parametric surface $\vec{r}(s,t) = [\sqrt{1+e^t}\cos(s)\,,\ \sqrt{1+e^t}\sin(s)\,,\ t\,]^T$ the grid curves with constant s are ellipses.

Solution:

Take $s = 0$ to get the curve $[\sqrt{1+e^t}, 0, t\,]^T$ which is the graph of a function of one variable in the xz plane.

10) T **F** There is a vector $\vec{v}$ with the property that $\vec{v} \times [1,1,1]^T = [0,0,1]^T$.

Solution:

Whatever the vector is, the right hand side would be perpendicular to $[1,1,1]^T$.

11) **T** F We can assign a value $f(0,0)$ such that the function $f(x,y) = (x^3+y^3)/(x^2+y^2)$ is continuous at $(0,0)$.

Solution:

Use polar coordinates to get $f = r^3(\cos^3(\theta) + \sin^3(\theta))/r^2 = r(\cos^3(\theta) + \sin^3(\theta))/r$.

12) T F The curvature of a curves $\vec{r}(t) = [t, t^2, t^3]^T$ and $\vec{R}(t) = [t^2, t^4, t^6]^T$ are the same at $t = 1$.

Solution:

Curvature is independent of the parametrization.

13) T F The curve given in spherical coordinates as $\phi = \pi/2, \rho = \pi/2$ is a circle.

Solution:

The result is zero

14) T F Two nonparallel planes with normal vectors $\vec{n}, \vec{m}$ intersect in a line parallel to $\vec{n} \times \vec{m}$.

Solution:

Make a picture

15) T F If $f(x, y) = x^2/3 + y^2$, then the graph of the function $f(x, y)$ is called a hyperbolic paraboloid.

Solution:

The graph is an elliptic paraboloid.

16) T F The equation $\rho \cos(\theta) \sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:

In spherical coordinates, we have $x = \rho \cos(\theta) \sin(\phi)$.

17) T F

The vector $[3, -2]^T$ in the two dimensional plane is perpendicular to the line $3x - 2y = 7$.

Solution:

It is the gradient $[1, 2]^T$.

18) T F

The volume of the parallelepiped spanned by the vectors $[1, 0, 0]^T$, $[0, 2, 0]^T$ and $[1, 1, 1]^T$ is 2.

Solution:

Compute the triple scalar product which is 2.

19) T F

If $\vec{r}(t)$ is a curve and $|\vec{r}\,'(t)| > 0$ and $|\vec{T}'| > 0$, we have $\vec{T}(t) \cdot (\vec{N}(t) \times \vec{B}(t)) = 1$.

Solution:

The three vectors are defined and are all perpendicular to each other and have length 1. They span a cube of volume 1.

20) T F

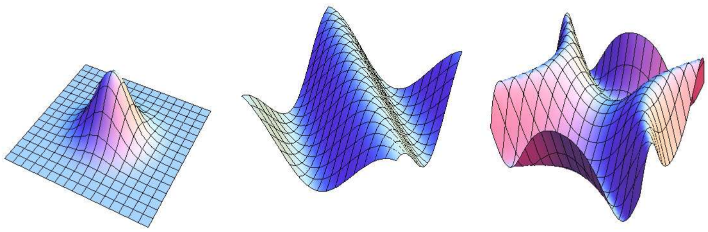
The arc lengths of $\vec{r}(t) = [t, t^2, t^3]^T$ and $\vec{R}(t) = [t^2, t^4, t^6]^T$ are the same for $0 \leq t \leq 1$.

Solution:

This is an important property of arc length.

Problem 2) (10 points)

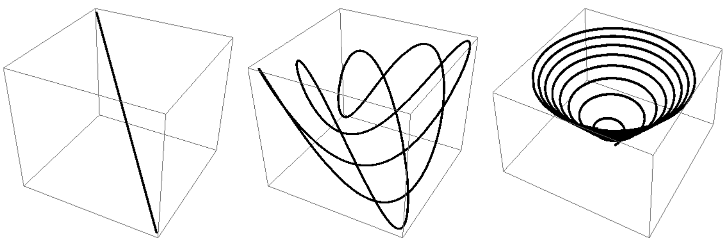
a) (2 points) Match the graphs $z = f(x, y)$ with the functions. Enter O, if there is no match. In each of the problems a) - d), each entry O,I,II,III appears exactly once.



I II III

Function $f(x, y) =$	O,I,II or III
$e^{-x^2-y^2}$	
$\cos(x + y)$	
$\sin(x^2 - y^2)$	
$x^4 + y^4$	

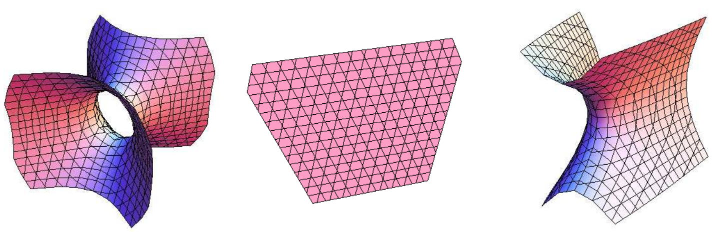
b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$[1 + t, 1 - t, t]^T$	
$[t\cos(t^2), t\sin(t^2), t]^T$	
$[t, t, \sin(t^3)]^T$	
$[\cos(3t), \sin(2t), \sin(5t)]^T$	

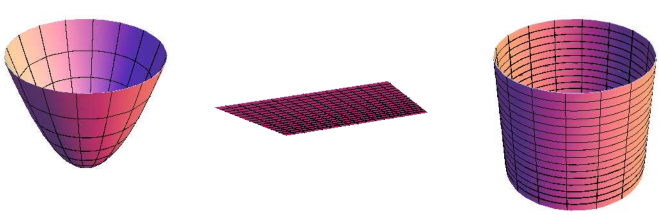
c) (2 points) Match the functions g with the level surface $g(x, y, z) = 1$. Enter O, where no match.



I II III

$g(x, y, z) =$	O, I,II,III
$(x - 1)^2 - y^2 + z^2 = 1$	
$(x - 1)^2 + y + z^2 = 1$	
$(x - 1) + y + z = 1$	
$(x - 1)^2 - y - z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.



I II III

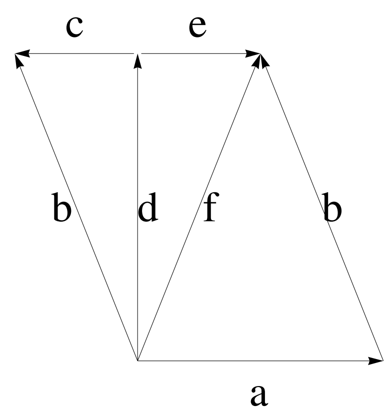
Parametriztion $\vec{r}(s, t) =$	O,I,II,III
$[s\cos(t), s\sin(t), s^2]^T$	
$[t - 1, s, s + t]^T$	
$[\cos(t), \sin(t), s]^T$	
$[s\cos(t), s\sin(t), s^2\sin(t)]^T$	

Solution:

a) I,II,III,O
b) I,III,O,II
c) I,O,II,III
d) I,II,III,O

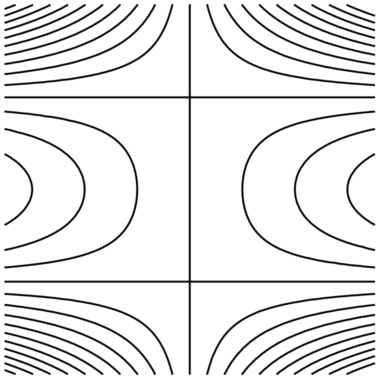
Problem 3) (10 points)

a) (7 points) Each of the vectors $a, b, c, d, e, f, 0$ (written without arrows for clarity) will appear in the blanks exactly once. As the picture indicates, you know $d \cdot e = d \cdot c = 0$.

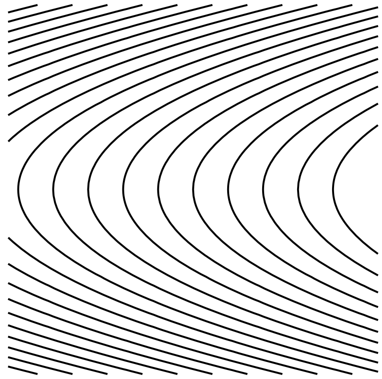


the vector	is equal to
$\text{proj}_d f$	
$f - d$	
$-2c$	
$d - c$	
$-e$	
$\text{proj}_d e$	
$d + c$	

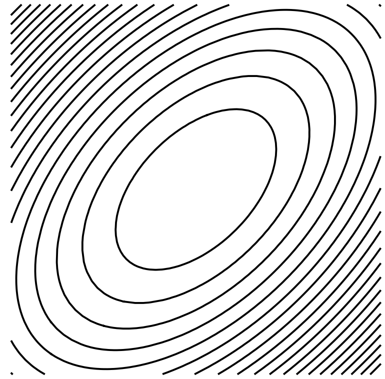
b) (3 points) Match the contour maps with the functions



I



II



III

Function $f(x,y) =$	Enter O,I,II or III
$y - x$	
$(y^2 - 1)x$	
$y^2 + x^2 - xy$	
$y^2 - x$	

Solution:

Its obviously a "deaf cob":

the vector	is equal to
$\text{proj}_{\vec{d}}\vec{f}$	$\vec{d}$
$\vec{f} - \vec{d}$	$\vec{e}$
$-2\vec{c}$	$\vec{a}$
$\vec{d} - \vec{c}$	$\vec{f}$
$-\vec{e}$	$\vec{c}$
$\text{proj}_{\vec{d}}\vec{e}$	$\vec{o}$
$\vec{d} + \vec{c}$	$\vec{b}$

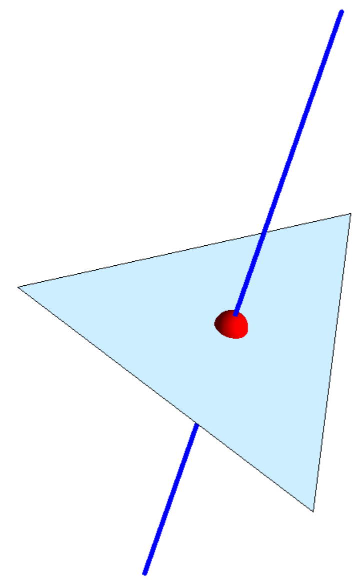
b) O,I,III,II

Problem 4) (10 points)

a) (4 points) The **center** of the triangle $A = (3,2,1), B = (1,1,1), C = (2,0,4)$ is the point $P = (A+B+C)/3 = (2,1,2)$. Find the line L perpendicular to the plane which contains A, B, C and which goes through P.

b) (3 points) Find the equation of the plane through A, B, C.

c) (3 points) Find the area of the triangle ABC.



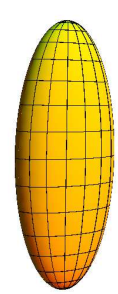
Solution:

The vectors $\vec{BA} = [2,1,0]^T, \vec{BC} = [1,-1,3]^T$ are in the plane. Their cross product $\vec{n} = [3,-6,-3]^T$ gives the direction normal to the plane as well as the direction of the line.
a) The equation of the line is $\vec{OP} + t\vec{n} = [2,1,2]^T + t[3,-6,-3]^T$.
b) The equation of the plane is $3x - 6y - 3z = -6$.
c) The area of the triangle is half the area of the parallelogram which is the length of the cross product divided by 2. This is $3\sqrt{6}/2$.

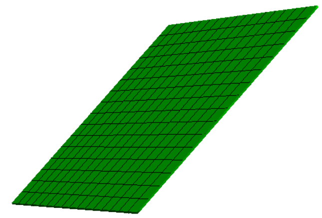
Problem 5) (10 points)

Complete the parametrizations:

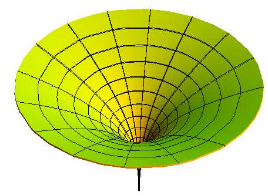
a) (3 points) $\vec{r}(u,v) = [2 + 3\cos(u)\sin(v), 3 + \sin(u)\sin(v), \boxed{}]^T$ parametrizes the ellipsoid $(x-2)^2/9 + (y-3)^2 + (z-5)^2/16 = 1$.



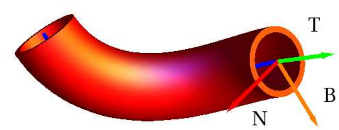
b) (2 points) $\vec{r}(u,v) = [u, v, \boxed{\quad}]^T$ parametrizes the plane $x + y + z = 1$.



c) (3 points) $\vec{r}(u,v) = [v^3\cos(u), \boxed{\quad}, v]^T$ parametrizes the surface of revolution $x^2 + y^2 = z^6$.



d) (2 points) $\vec{r}(u,v) = \vec{r}(v) + \cos(u)\vec{N}(v) + \sin(u)\boxed{\quad}$ parametrizes a tube around a curve $\vec{r}(v)$ which has unit tangent vector $\vec{T}(v)$, normal vector $\vec{N}(v)$ and binormal vector $\vec{B}(v)$.



Solution:

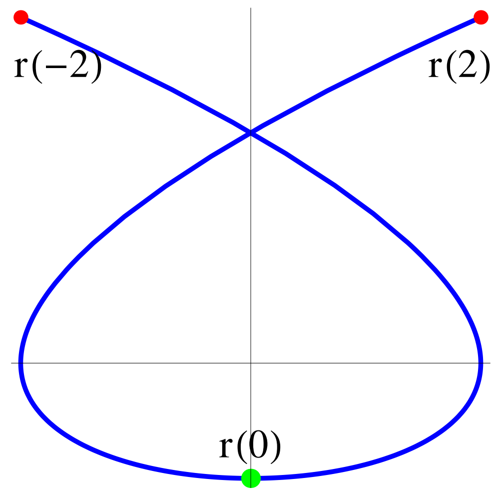
a) This is an ellipsoid centered at $(2, 3, 5)$ which is deformed. The last entry is $\boxed{5 + 4\cos(v)}$.
b) This is a plane. Solve for $z = 1 - x - y$ and use u, v to get $\boxed{1 - u - v}$.
c) This is a surface of revolution and we have $\boxed{v^3\sin(u)}$.
d) Since the tube circular and the grid curves with constant v are circles, we can use N, B to draw the circle. This is almost completed. W only have to fill in $\boxed{\vec{B}(v)}$. This example is actually one of the main motivations to use the TNB frame. It allows to draw beautiful tubes around a given curve.

Problem 6) (10 points)

We look at the parametrized curve

$$\vec{r}(t) = [\frac{t^3}{3} - t, t^2 - 1, 0]^T$$

whose image you see in the picture showing it in the xy plane for $-2 \leq t \leq 2$.



a) (3 points) Find the velocity $\vec{r}\,'(t)$, the acceleration $\vec{r}\,''(t)$ and speed $|\vec{r}\,'(t)|$.

b) (2 points) Evaluate this at $t = 0$ to get $\vec{r}\,'(0), \vec{r}\,''(0)$ and $|\vec{r}\,'(0)|$.

c) (2 points) Find the curvature $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$ at $(0, -1, 0)$.

d) (3 points) Find the arc length of the curve $\vec{r}(t)$ from $-2 \leq t \leq 2$.

Solution:

a) $\vec{r}\ '(t) = [t^2 - 1, 2t, 0]^T$. $\vec{r}\ ''(t) = [2t, 2, 0]^T$ and $|[t^2 - 1, 2t, 0]^T| = \sqrt{t^4 + 2t + 1} = \sqrt{(t^2+1)^2} = t^2 + 1$.
b) $\vec{r}\,'(0) = [-1, 0, 0]^T, \vec{r}\,''(0) = [0, 2, 0]^T$ and $|\vec{r}\,'(0) = 1$.
c) Since the speed is 1, the curvature is

$$|[-1, 0, 0]^T \times [0, 2, 0]^T| = |[0, 0, -2]^T| = 2$$

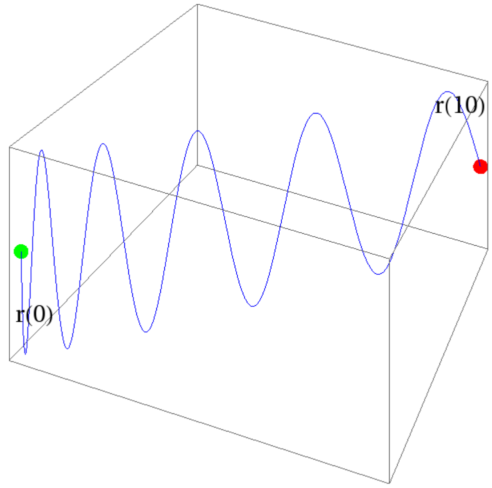
d) The arc length is $\int_{-2}^{2} |\vec{r}\,'(t)|\ dt = \int_{-2}^{2} t^2 + 1\ dt = t^3/3 + t|_{-2}^{2} = 28/3$.

Problem 7) (10 points)

a) (4 points) We know $\vec{r}\ ''(t) = [1, 2, \pi^2 \sin(\pi t)]^T$ and the initial velocity $\vec{r}\,'(0) = [1, 0, -\pi]^T$. Find $\vec{r}\,'(t)$.

b) (3 points) Assume we know also $\vec{r}(0) = [0, 0, 10]^T$. Find $\vec{r}(10)$.

c) (3 points) What is the projection of $\vec{r}\ '(10)$ onto $[1, 1, 0]^T$?



Solution:

a) Integrate to get

$$\vec{r}\,'(t) = [t, 2t, -\pi \cos(\pi t)]^T + [c_1, c_2, c_3]^T \ .$$

Comparing the initial velocity gives the constants and so

$$\vec{r}'(t) = [1 + t, 2t, -(\pi \cos(\pi t))]^T \ .$$

b) Integrate again and compare coefficients to get

$$\vec{r}(t) = t + t^2/2, t^2, 10 - \sin(\pi t) \ .$$

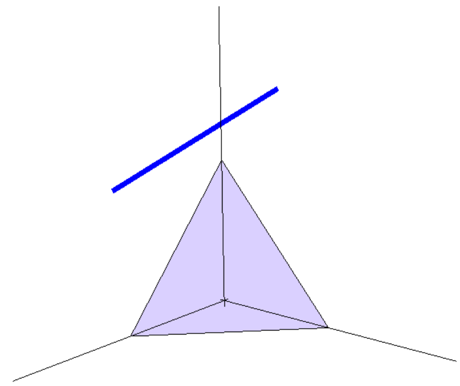
we have $r(10) = [60, 100, 10]^T$.
c) We have $\vec{r}\ '(10) = [11, 20, -\pi]^T$. The vector projection is $\vec{r}\ '(10) \cdot [1, 1, 0]^T/2 = (31/2)[1, 1, 0]^T$.

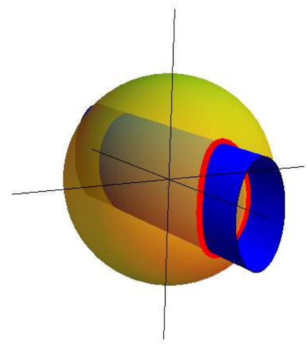
Problem 8) (10 points)

a) (5 points) Find the distance between the plane $x + y + z = 1$ and the line

$$x - 1 = \frac{(y-1)}{-2} = z - 1$$

which is parallel to the plane.
(You do not have to check that it is parallel).





b) (5 points) The intersection of the cylinder $4x^2+z^2 = 1$ with the sphere centered at $(0,0,0)$ with radius $\rho = \sqrt{2}$ cuts out two curves. Parametrize the curve which contains the point $(0,1,1)$.

Solution:

a) Choose a point on the plane $P = (1,0,0)$ and a point $Q = (1,1,1)$ on the line and compute the distance from P to the plane We have $\vec{PQ} = [0,1,1]^T$ and

$$d = \frac{|[0,1,1]^T \cdot [1,1,1]^T|}{|[1,1,1]^T|} = 2/\sqrt{3} \ .$$

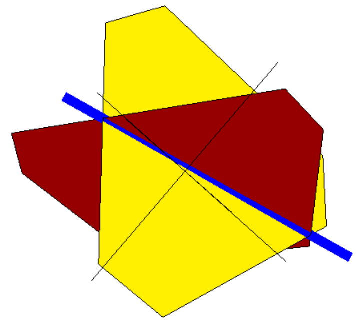
b) First parametrize the ellipse in the xz-plane with $\vec{r}(t) = [\cos(t)/2, ..., \sin(t)]^T$ where we do not know y yet. The sphere has the equation $x^2 + y^2 + z^2 = 2$ and we can solve for y and get the parametrization

$$\vec{r}(t) = [\cos(t)/2, \sqrt{2 - \cos^2(t)/4 - \sin^2(t)}, \sin(t)]^T \ .$$

Problem 9) (10 points)

a) (5 points) Find a parametrization of the intersection line L of the two planes

$$\begin{aligned} 2x - 2y + z &= 1 \,, \\ x + y + z &= 1 \,. \end{aligned}$$



b) (5 points) Find the symmetric equation for the line M parallel to the line L computed in a) which passes through $(1, 2, 3)$.

Solution:

a) A point in the intersection of the plane is $P = (0, 0, 1)$. The cross product between the normal vectors is $[-3, -1, 4]^T$. The parametrization of the line is $\vec{r}(t) = [0, 0, 1]^T + t[-3, -1, 4]^T$.
b) Now translate the line through $(1, 2, 3$ to get $\vec{r}(t) = [1, 2, 3]^T + t[-3, -1, 4]^T$ which has the symmetric equations

$$\frac{x-1}{(-3)} = \frac{y-2}{(-1)} = \frac{z-3}{4} \,.$$