

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1,2 and 8, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) No justifications are needed.

- 1) T F The distance between two points P and Q is equal to length of the vector \vec{QP} .

Solution:

They both are the square root of the sum of the squares of the differences of the coefficients respectively the components.

- 2) T F The symmetric equations of $\vec{r}(t) = [1 + t, 2 + t, 3 + t]^T$ is $x = y = z$.

Solution:

The correct symmetric equation is $x - 1 = y - 2 = z - 3$.

- 3) T F The arc length of a circle with constant curvature 6 is $\pi/3$.

Solution:

The radius of the circle must be $1/6$. Multiply this by 2π to get the arc length.

- 4) T F The surface $x^2 + y^2 + 4y = z^2 + 2z$ is a cone.

Solution:

Complete the square. It is a hyperboloid.

- 5) T F A vector is characterized as an entity which has magnitude and direction.

Solution:

Yes, this is often used as a characterization, but it is wrong. The zero vector has no direction. We pointed out in class also that this is a bad definition because it is done without having defined magnitude and direction. Both concepts magnitude and direction are defined after vectors are defined. We also pointed out in the notes that a Movie would qualify as it is an entity which has magnitude (length) and direction (there is hardly any movie without director).

- 6) T F The Cauchy-Schwartz inequality tells that $|\vec{v} \cdot \vec{w}| \leq |\vec{v} + \vec{w}|$.

Solution:

It is not plus on the right hand side but times.

- 7) T F The acceleration of $\vec{r}(t) = [t, t, t]^T$ is zero everywhere.

Solution:

Indeed, the second derivative is $[0, 0, 0]^T$.

- 8) T F If $\vec{v} = \vec{PQ} = [2, 1, 1]^T$ then $|\vec{v}| = d(P, Q)$.

Solution:

By definition

- 9) T F The surface $x^2 - y^2 + z^2 - 2x = 3$ is a one-sheeted hyperboloid.

Solution:

Complete the square

- 10) T F If $\vec{v} \cdot \vec{w}$ is positive, then the angle between \vec{v} and \vec{w} is larger than 90° .

Solution:

Use the cos formula. Positive means actually that the angle is acute.

- 11) T F There exists a differentiable function for which the level curves $f(x, y) = 1$ and $f(x, y) = 0$ intersect in a point.

Solution:

A function can not take the value 1 and 0 at the same time.

- 12) T F The equation $[1, 0, 1]^T \times \vec{x} = \vec{x}$ has only the zero vector \vec{x} as solution.

Solution:

Yes, since \vec{x} must be perpendicular to \vec{x} .

- 13) T F The curvature of $\vec{r}(t) = [t - 1, 1 - t, t]^T$ is 0 if $t = 1$.

Solution:

It is a line

- 14) T F The line $\vec{r}(t) = t[3, 4, 5]^T$ hits the plane $-4x + 3y = 0$ in a right angle.

Solution:

The normal vector is $-4, 3, 0$.

- 15) T F The point given in spherical coordinates as $\rho = 3, \phi = \pi/2, \theta = \pi$ is on the z -axes.

Solution:

It is on the equator

- 16) T F Given three vectors \vec{u}, \vec{v} and \vec{w} , then $|(\vec{u} \cdot \vec{v})\vec{w}| \leq |\vec{u}||\vec{v}||\vec{w}|$.

Solution:

Use the cos identity for the length of the dot product

- 17) T F The surface given in spherical coordinates as $\rho \cos(\theta) = \rho^2$ is a sphere.

Solution:

Look at the traces

- 18) T F The arc length of the curve $[\sin(t), 1, \cos(t)]^T$ from $t = 0$ to $t = 2\pi$ is equal to 2π .

Solution:

It is a circle

- 19) T F The surface parametrized by $\vec{r}(u, v) = [u^3 + v^3, u^3 - v^3, u^3]^T$ is a plane.

Solution:

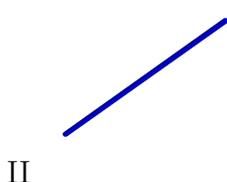
$x + y = 2z$

- 20) T F If the cross product of a vector \vec{v} with a vector \vec{w} is parallel to \vec{v} then the dot product between \vec{v} and \vec{w} is zero.

Total

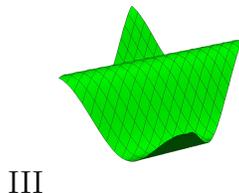
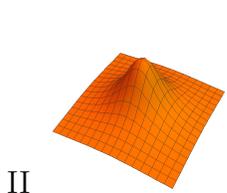
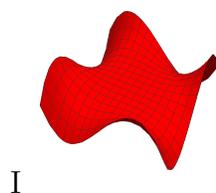
Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the curves. Enter O, if there is no match.



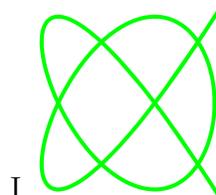
Space curve $\vec{r}(t) =$	Enter O,I,II or III
$[\cos(t), \sin(t), t]^T$	
$[t, t \sin(t), t \cos(t)]^T$	
$[1, t^2 \cos(t), t^2 \cos(t)]^T$	
$[t, t^2, t^3]^T$	
$[1 + t, t, t]^T$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.



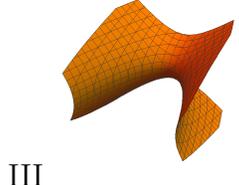
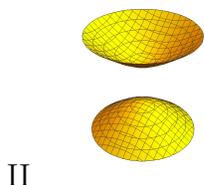
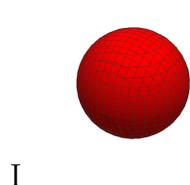
Function $f(x, y) =$	Enter O,I,II or III
$x^3y - y^3x$	
$x^2 + y^2$	
$\cos(x - y)$	
y^3	
$\sin(1/(1 + x^2 + y^2))$	

c) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



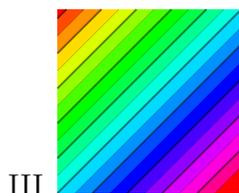
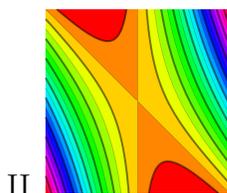
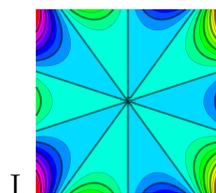
Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = [t, t^4]^T$	
$\vec{r}(t) = [\cos(4t), \sin(5t)]^T$	
$\vec{r}(t) = [t \cos(5t), t \sin(5t)]^T$	
$\vec{r}(t) = [t^2, t]^T$	
$\vec{r}(t) = [2t, 3t]^T$	

d) (2 points) Match functions g with level surface $g(x, y, z) = 1$. Enter O, if no match.



Function $g(x, y, z) = 1$	Enter O, I,II or III
$x^2 + y^2 - z^2 = 1$	
$x^2 + y - z^2 = 1$	
$x + y - z^2 = 1$	
$x^2 + y^2 + z^2 = 1$	
$x^2 + y^2 - z^2 = -1$	

e) (2 points) Match the contour maps to a function $f(x, y)$. Enter O if no match.



$f(x, y) =$	Enter O,I,II or III
$xy + x^2$	
$x - y$	
x^2	
$x^5 - 10x^3y^2 + 5xy^4$	

Solution:

- a) I,III,0,0,II
- b) I,0,III,0,II
- c) 0,I,III,II,0
- d) 0,III,0,I,II
- e) II,III,0,I

Problem 3) (10 points)

Most problems in the multivariable textbook of **Willard Gibbs** and **Edwin Wilson** from 1901 at Yale were proof based. (This book by the way is the prototype of all multi variable textbooks since). Today we like more to compute with concrete vectors. Define $\vec{v} = [4, 3, 2]^T$ and $\vec{w} = [2, 3, 4]^T$. Let α be the angle between \vec{v} and \vec{w} . No, lets even be more basic and compute some **numbers**:

- a) (2 points) Compute $\vec{v} \cdot \vec{w}$.
- b) (2 points) Compute $|\vec{v}| \cdot |\vec{w}|$.
- c) (2 points) Compute $\cos(\alpha)$.
- d) (2 points) Compute $|\vec{v} \times \vec{w}|$.
- e) (2 points) Compute $\sin(\alpha)$.

Solution:

- a) 25
- b) 29
- c) $25/29$
- d) $\sqrt{216}$
- e) $\sqrt{216}/29$.

Problem 4) (10 points)

EXERCISES ON CHAPTER II

Prove the following reduction formulæ

$$1. \mathbf{A} \times \{\mathbf{B} \times (\mathbf{C} \times \mathbf{D})\} = [\mathbf{A} \mathbf{C} \mathbf{D}] \mathbf{B} - \mathbf{A} \cdot \mathbf{B} \mathbf{C} \times \mathbf{D} \\ = \mathbf{B} \cdot \mathbf{D} \mathbf{A} \times \mathbf{C} - \mathbf{B} \cdot \mathbf{C} \mathbf{A} \times \mathbf{D}.$$

$$2. [\mathbf{A} \times \mathbf{B} \ \mathbf{C} \times \mathbf{D} \ \mathbf{E} \times \mathbf{F}] = [\mathbf{A} \mathbf{B} \mathbf{D}] [\mathbf{C} \mathbf{E} \mathbf{F}] - [\mathbf{A} \mathbf{B} \mathbf{C}] [\mathbf{D} \mathbf{E} \mathbf{F}] \\ = [\mathbf{A} \mathbf{B} \mathbf{E}] [\mathbf{F} \mathbf{C} \mathbf{D}] - [\mathbf{A} \mathbf{B} \mathbf{F}] [\mathbf{E} \mathbf{C} \mathbf{D}] \\ = [\mathbf{C} \mathbf{D} \mathbf{A}] [\mathbf{B} \mathbf{E} \mathbf{F}] - [\mathbf{C} \mathbf{D} \mathbf{E}] [\mathbf{A} \mathbf{B} \mathbf{F}]$$

$$3. [\mathbf{A} \times \mathbf{B} \ \mathbf{B} \times \mathbf{C} \ \mathbf{C} \times \mathbf{A}] = [\mathbf{A} \mathbf{B} \mathbf{C}]^2.$$

$$4. [\mathbf{P} \mathbf{Q} \mathbf{R}] (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{P} \cdot \mathbf{A} & \mathbf{P} \cdot \mathbf{B} & \mathbf{P} \\ \mathbf{Q} \cdot \mathbf{A} & \mathbf{Q} \cdot \mathbf{B} & \mathbf{Q} \\ \mathbf{R} \cdot \mathbf{A} & \mathbf{R} \cdot \mathbf{B} & \mathbf{R} \end{vmatrix}.$$

$$5. \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0.$$

$$6. [\mathbf{A} \times \mathbf{P} \ \mathbf{B} \times \mathbf{Q} \ \mathbf{C} \times \mathbf{R}] + [\mathbf{A} \times \mathbf{Q} \ \mathbf{B} \times \mathbf{R} \ \mathbf{C} \times \mathbf{P}] \\ + [\mathbf{A} \times \mathbf{R} \ \mathbf{B} \times \mathbf{P} \ \mathbf{C} \times \mathbf{Q}] = 0.$$

7. Obtain formula (34) in the text by expanding $[(\mathbf{A} \times \mathbf{B}) \times \mathbf{P}] \cdot [\mathbf{C} \times (\mathbf{Q} \times \mathbf{R})]$



Busy constructions are going on at Harvard this summer. Assume a crane is given by a line

$$L : \vec{r}(t) = [1 + t, 2, t]^T .$$

John Harvard (rsp. **Sherman Hoar** if you know about the three lies), watches the crane carefully from his location

$$P = (3, 4, 5) .$$

He is very concerned to have his shiny shoe scratched or worse, to have his head knocked off and give us an opportunity to replace the impostor with the real John Harvard. What is the distance between the point P and the line L ?



Solution:

$$\vec{v} = [1, 0, 1]^T . \quad \vec{PQ} = [2, 2, 5]^T . \quad d = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|} = \sqrt{17/2} .$$

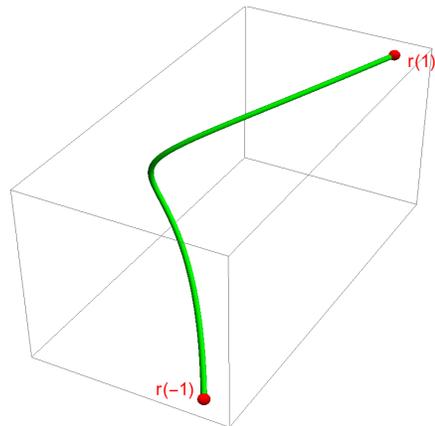
Problem 5) (10 points)

a) (7 points) Find the arc length of the curve

$$\vec{r}(t) = [1 + \sqrt{3}t^2, 1 + 2t^3, 1 + t]^T$$

from $t = -1$ to $t = 1$.

b) (3 points) Evaluate the integral $|\int_{-1}^1 \vec{r}'(t) dt|$.



Solution:

a) $\int_{-1}^1 \sqrt{12t^2 + 36t^4 + 1} dt = \int_{-1}^1 6t^2 + 1 dt = 6$.

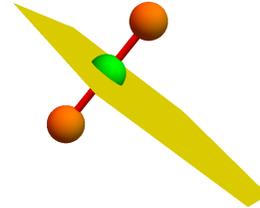
b) The fundamental theorem of calculus assures $\int_{-1}^1 \vec{r}'(t) dt = |\vec{r}(1) - \vec{r}(-1)| = \sqrt{20}$

Problem 6) (10 points)

(7 points) Find the equation

$$ax + by + cz = d$$

for the plane which passes through the midpoint $M = (P + Q)/2$ of the two points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$. We want the plane to have the property that the line through PQ hits the plane perpendicular at M .



(3 points) What is the distance between P and the plane constructed in a)?

Solution:

- a) $x + y + z = 9$. The vector $[2, 2, 2]^T$ is perpendicular. This gives $[a, b, c]^T$.
b) $d(M, P) = \sqrt{3}$. Just take the distance between P and M .

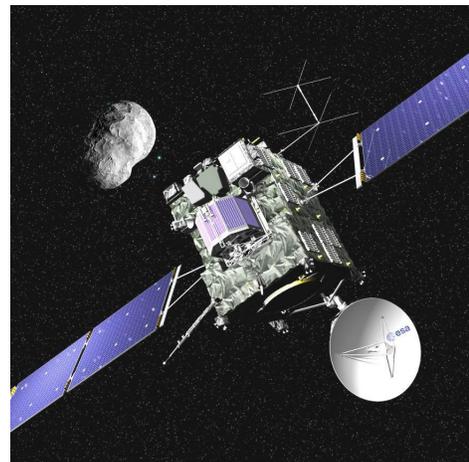
Problem 7) (10 points)

This September 2016, the **Rosetta** spacecraft will make a suicide plunge onto the comet comet Churyumov-Gerasimenk and gather a few last data. Assume its path is $\vec{r}(t) = [t, 2t, 1 - t^2]^T$.

- a) (2 points) Find the velocity at $t = 1$.
b) (2 points) Find the acceleration vector at $t = 1$.
c) (2 points) Find the jerk vector $\vec{r}'''(t)$ at $t = 1$.
d) (2 points) Give the unit tangent vector at $t = 1$.
e) (2 points) Compute the curvature

$$\frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

at $t = 1$.



Solution:

- a) $[1, 2, -2]^T$.
- b) $[0, 0, -2]^T$.
- c) $[0, 0, 0]^T$.
- d) $[1, 2, -2]^T/3$.
- e) The cross product is $[-4, 2, 0]$ which has length $\sqrt{20}$. The curvature is $\sqrt{20}/27$.

Problem 8) (10 points)

We enjoy the summer and ask in a local restaurant for a refreshment. In each of the ordered items, give a **surface parametrization** of the form

$$\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]^T .$$

As indicated, we can use also other variables. Your task is to fill in the three parametrization functions in each case, using the variables provided.

- a) (2 points) Parametrize the **lemonade glass**
 $x^2 + y^2 = 1$.

$$\vec{r}(\theta, z) = \left[\boxed{}, \boxed{}, \boxed{} \right]^T .$$

- b) (2 points) Parametrize the **sorbet glass** $x^2 + y^2 = z^2$.

$$\vec{r}(\theta, z) = \left[\boxed{}, \boxed{}, \boxed{} \right]^T .$$

- c) (2 points) Parametrize the **lemon** surface
 $x^2 + y^2 = \sin(z)$.

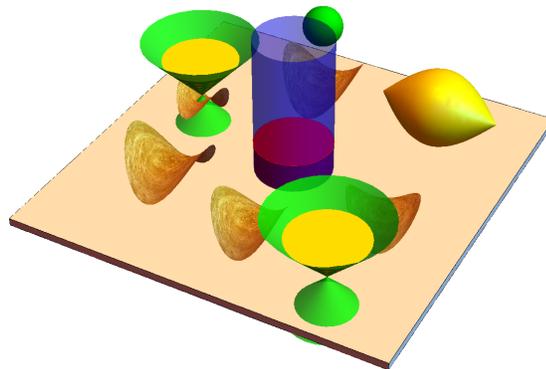
$$\vec{r}(\theta, z) = \left[\boxed{}, \boxed{}, \boxed{} \right]^T .$$

- d) (2 points) Parametrize one of the **chips** $z = x^2 - y^2$.

$$\vec{r}(x, y) = \left[\boxed{}, \boxed{}, \boxed{} \right]^T .$$

- e) (2 points) Parametrize the **lime** $x^2 + y^2 + (z - 3)^2 = 1$.

$$\vec{r}(\theta, \phi) = \left[\boxed{}, \boxed{}, \boxed{} \right]^T .$$



Solution:

- a) $[\cos(\theta), \sin(\theta), z]^T$.
- b) $[z \cos(\theta), z \sin(\theta), z]^T$.
- c) $[\sqrt{\sin(z)} \cos(\theta), \sqrt{\sin(z)} \sin(\theta), z]^T$.
- d) $[x, y, x^2 - y^2]^T$.
- e) $[\sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), 3 + \cos(\phi)]^T$.

Problem 9) (10 points)

The **Juno space craft** has arrived at Jupiter on July 4. It is an exciting and dangerous journey. For example: since the magnetic field of Jupiter is 20'000 times stronger than the earth's magnetic field, it required the electronics to be shielded from radiation. The acceleration on the space craft is given by

$$\vec{r}''(t) = [t, t^2, 1 - t]^T .$$

We know that the position at $t = 0$ is given by $\vec{r}(0) = (3, 4, 5)$ when $t = 0$. Additionally, we know that its velocity at time $t = 1$ is $\vec{r}'(1) = [1, 0, 0]^T$. (Note $t = 1$ for velocity measurement and $t = 0$ for initial position are different!)

Find the position of the space craft at time $t = 3$.

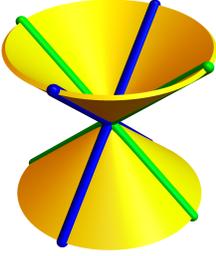
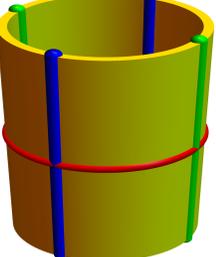
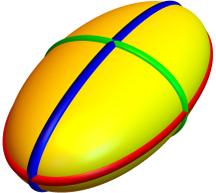
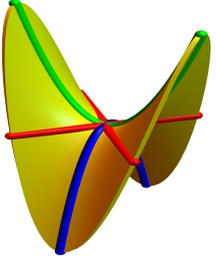
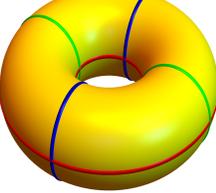


Solution:

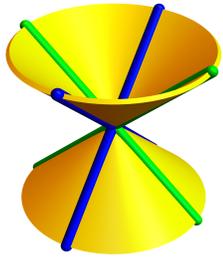
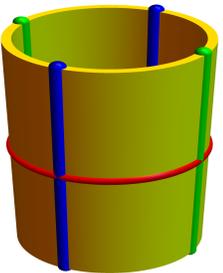
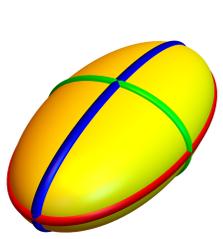
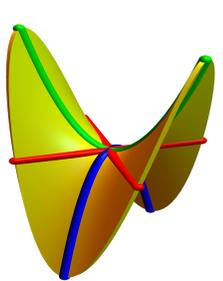
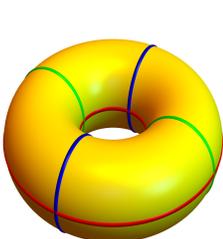
Integrate a first time $\vec{r}'(t) = [t^2/2 + 1/2, t^3/3 - 1/3, t - t^2/2 - 1/2]^T$. Then integrate again $\vec{r}(t) = [t^3/6 + t/2 + 3, t^4/12 - t/3 + 4, t^2/2 - t^3/6 - t/2 + 5]^T$. At $t = 3$, this is $[9, 39/4, 7/2]^T$.

Problem 10) (10 points)

The 3D printing venture "Math-Candy" (math-candy.com) asks you to do some product development. In each of the 5 following parametrizations, two entries are still missing, each entry being worth one candy (1 point).

a)		<p>The surface</p> $x^2 + y^2 = z^2$ <p>is parametrized by</p> $\vec{r}(\theta, z) = [\dots, \dots, z]^T$
b)		<p>The surface</p> $x^2 + y^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, z) = [\dots, \dots, z]^T$
c)		<p>The surface</p> $2(x - 1)^2 + (y - 5)^2 + 4z^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, \phi) = [\dots, \dots, \cos(\phi)/2]^T$
d)		<p>The surface</p> $x^2 - y^2 = z$ <p>is parametrized by</p> $\vec{r}(x, y) = [\dots, \dots, x^2 - y^2]^T$
e)		<p>The surface</p> $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, \phi) = [(2 + \cos(\phi)) \cos(\theta), \dots, \dots]^T$

Solution:

a)		<p>The surface</p> $x^2 + y^2 = z^2$ <p>is parametrized by</p> $\vec{r}(\theta, z) = [z \cos(\theta), z \sin(\theta), z]^T$
b)		<p>The surface</p> $x^2 + y^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, z) = [\cos(\theta), \sin(\theta), z]^T$
c)		<p>The surface</p> $2(x - 1)^2 + (y - 5)^2 + 4z^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, \phi) = [\frac{[\sin(\theta) \cos(\theta) + 1]}{\sqrt{2}}, 5 + \sin(\theta) \sin(\phi), \cos(\phi)/2]^T$
d)		<p>The surface</p> $x^2 - y^2 = z$ <p>is parametrized by</p> $\vec{r}(x, y) = [x, y, x^2 - y^2]^T$
e)		<p>The surface</p> $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$ <p>is parametrized by</p> $\vec{r}(\theta, \phi) = [(2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi)]^T$