

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

- 1) ☒ T ☐ F The lines $\vec{r}_1(t) = [t, t, -t]^T$ and $\vec{r}_2(t) = [1 + t, 1 + t, 1 - t]^T$ do not intersect.

Solution:

Indeed, they are parallel.

- 2) ☐ T ☒ F The quadratic surface $x^2 - y^2 = z^2$ is a hyperbolic paraboloid.

Solution:

No, it is a cone.

- 3) ☒ T ☐ F If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}'(t) \neq 0$ everywhere, then they span a parallelepiped of volume 1.

Solution:

The three vectors are perpendicular and have length 1. The parallelepiped is a cube of volume 1.

- 4) ☒ T ☐ F If $\vec{u} \cdot \vec{v} = 0$, then $\text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.

Solution:

The two vectors are then perpendicular.

- 5) ☒ T ☐ F There is a vector field $\vec{F}(x, y)$ which has the property $\text{curl}(\vec{F}) = -\text{div}(\vec{F})$, where $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$.

Solution:

Take an incompressible irrotational field. Any constant field works

- 6) ☐ T ☒ F The acceleration vector $\vec{r}''(t) = [x(t), y(t), z(t)]^T$ is always a unit vector if the velocity vector $\vec{r}'(t)$ is a unit vector.

Solution:

Take $\vec{r}(t) = [t^3/3, 0, 0]^T$ at time $t = 1$ for example

- 7)

T	F
---	---

 The grid curves $t \rightarrow \vec{r}(t, \phi)$ with fixed $0 < \phi < \pi$ for the standard parametrization of the unit sphere have curvature $1/\sin(\phi)$.

Solution:

The radius is $\sin(\phi)$.

- 8)

T	F
---	---

 Any smooth function $f(x, y)$ has a local maximum somewhere in the plane.

Solution:

$x^2 - y^2$ does not have a maximum, nor a minimum.

- 9)

T	F
---	---

 The linearization $L(x, y)$ of constant function $f(x, y) = 3$ is $L(x, y) = 3$.

Solution:

The linearization of linear function is the same function

- 10)

T	F
---	---

 A gradient field is incompressible: it satisfies $\text{div}(F) = 0$ everywhere.

Solution:

It is irrotational not necessarily incompressible. An example is $\vec{F}(x, y, z) = [x, y, z]^T$.

- 11)

T	F
---	---

 If $f(x, y)$ has a maximum under the constraint $g(x, y) = 1$, then $\nabla f = [0, 0]^T$ at this point.

Solution:

Critical points under constraints are not necessarily critical points without constraint.

- 12)

T	F
---	---

 Assume a vector field $\vec{F}(x, y, z)$ is the curl of a vector field $\vec{G}$ then the flux of the field F through the ellipsoid $x^2 + y^2 + 5z^2 \leq 1$ is zero.

Solution:

This follows both from Stokes theorem (no boundary curve) or the divergence theorem because $\text{div}(\text{curl}(F)) = 0$.

- 13)

T

F

 If the divergence of a field $\vec{F}$ are zero everywhere, then any line integral along a closed curve is zero.

Solution:

It is the curl which matters, when we look at line integrals along closed loops.

- 14)

T

F

 The gradient of the divergence of a field is always the zero field.

Solution:

Take $f(x, y, z) = [x^2, y^2, z^2]^T$ for which the gradient of the divergence is $[2, 2, 2]^T$

- 15)

T

F

 The vector field $\vec{F}(x, y, z) = [x^2, y^2, z^3]^T$ is a gradient field.

Solution:

Yes, the potential is $f(x, y, z) = x^3/3 + y^3/3 + z^4/4$.

- 16)

T

F

 The volume of a solid can be computed as the flux of the field $[0, y, 0]^T$ through the boundary surface.

Solution:

The field has constant divergence 1 so that by the divergence theorem the flux of the field through the boundary surface is equal to the volume.

- 17)

T

F

 The curvature of a line is zero.

Solution:

Yes, there is no change of the unit tangent vector. You can also see it from the fact that the velocity and acceleration are parallel. —

- 18) ☒ T ☐ F The distance between the unit sphere centered at $(0, 0, 0)$ and the plane $z = 5$ is equal to 4.

Solution:

It is by 1 smaller than the distance of the origin to the plane.

- 19) ☐ T ☒ F The partial differential equation $u_t = u_x$ is called heat equation.

Solution:

It is the transport equation

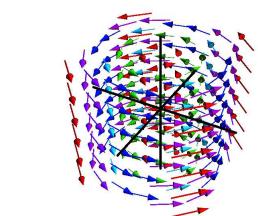
- 20) ☐ T ☒ F The point $(1, -1, \sqrt{2})$ in spherical coordinates is $(\rho, \phi, \theta) = (2, \pi/4, 3\pi/2)$.

Solution:

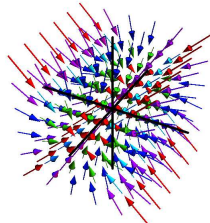
The first two entries are correct but the θ angle does not match. It would be $7\pi/4$.

Problem 2) (10 points) No justifications are necessary.

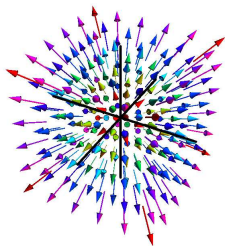
a) (4 points) Match the objects with the definitions.



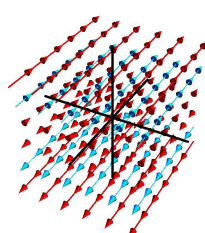
1



2



3

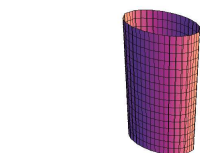


4

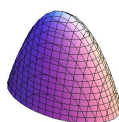
enter 1-4	vector field
	$\vec{F}(x, y, z) = [x, y, z]^T$
	$\vec{F}(x, y, z) = [-y, x, 0]^T$
	$\vec{F}(x, y, z) = [0, z, 0]^T$
	$\vec{F}(x, y, z) = [-x, 0, -z]^T$

Solution:

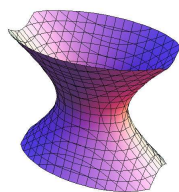
b) (3 points) Match the surfaces with their names: (put O if no match)



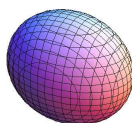
1



2



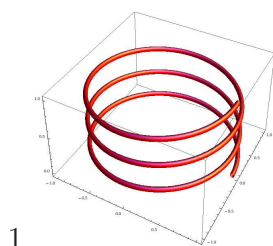
3



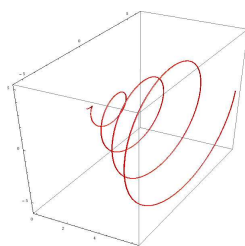
4

enter 1-4	surface
	$x^2 + y^2 + 3z = 0$
	$x^2 + y^2 - 3z^2 = 1$
	$x^2 + y^2 + 3z^2 = 1$
	$x^3 + 3y^2 = 1$

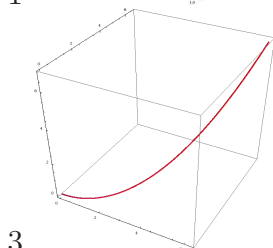
c) (3 points) Match the space curves



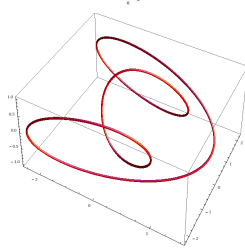
1



2



3



4

1-4	parametrized curve
	$\vec{r}(t) = [t, t^2, t^3]^T$
	$\vec{r}(t) = [\cos(3t), \sin(3t), t]^T$
	$\vec{r}(t) = [(2 + \cos(t)) \cos(3t), (2 + \cos(t)) \sin(t), \sin(3t)]^T$
	$\vec{r}(t) = [t, t \cos(3t), t \sin(3t)]^T$

Solution:

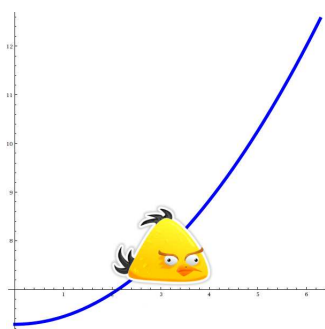
a) 3,1,4,2

b) 2,3,4 0, The $x^3 + 3y^2 = 1$ is cylindrical but not bounded in the xy plane.

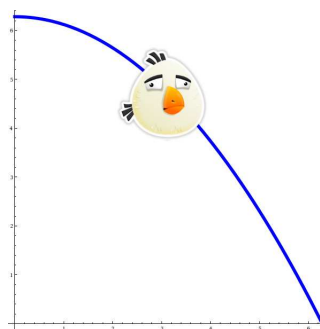
c) 3,1,4,2

Problem 3) (10 points) No justifications are necessary

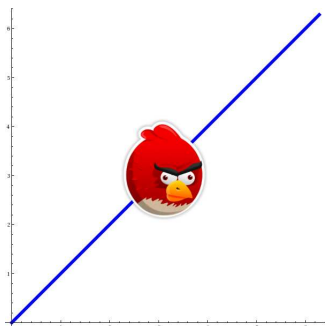
a) (5 points) We watch "angry birds" attacking on curves with acceleration $\vec{r}''(t)$. (The pictures show the xz - planes and the birds start with a constant velocity $[1, 0, 0]^T$.) Match the displayed curves $\vec{r}(t)$ with the formulas for accelerations.



1

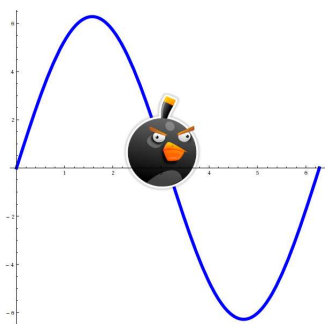


2

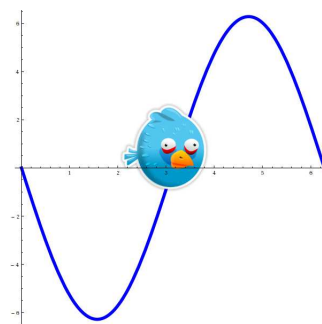


3

acceleration	enter curve 1-5
$\vec{r}''(t) = [0, 0, \sin(t)]^T$	
$\vec{r}''(t) = [0, 0, -10]^T$	
$\vec{r}''(t) = [0, 0, 10]^T$	
$\vec{r}''(t) = [0, 0, -\sin(t)]^T$	
$\vec{r}''(t) = [0, 0, 0]^T$	



4



5

b) (5 points) Match the formulas: (put O if no match)

label	formula
A	$\vec{r}'(t)$
B	$\int_0^1 \vec{r}'(t) dt$
C	$\vec{r}'(t)/ \vec{r}'(t) $
D	$\vec{T}'(t)/ \vec{T}'(t) $
E	$ \vec{r}'(t) \times \vec{r}''(t) / \vec{r}'(t) ^3$

expression	enter A-E
Curvature	
Unit tangent vector	
Unit normal vector	
Velocity	
Arc length	

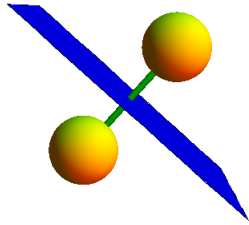
Solution:

a) 5,2,1,4,3

A difficulty had been the curves with the $\sin(t)$ entries. Integrating this twice gives $-\sin(t)$. The $\sin(t)$ z-acceleration produces the curve $r(t)$ with $-\sin(t)$ in the z coordinate.

b) E,C,D,A,B.

Problem 4) (10 points)



a) (5 points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y - 1)^2 + z^2 = 1$, $(x - 5)^2 + y^2 + z^2 = 1$.

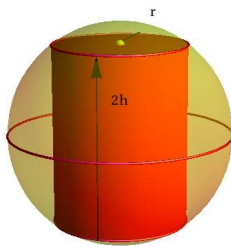
b) (5 points) Find the plane perpendicular to the line L for which the distances to the spheres are the same.

Solution:

a) The centers of the sphere are $(0, 1, 0)$ and $(5, 0, 0)$. A parametrization of the line connecting these two points is $\vec{r}(t) = [0, 1, 0]^T + t[5, -1, 0]^T$. It can be written also as $\vec{r}(t) = [5t, 1 - t, 0]^T$.

b) The midpoint between the two points is $(5/2, 1/2, 0)$. Because the normal vector to the plane is $[5, -1, 0]^T$, the plane has the equation $5x - y = d$ where the constant d can be obtained by plugging in the midpoint. The answer is $5x - y = 12$.

Problem 5) (10 points)



Johannes Kepler asked which cylinder of radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve his problem, use the Lagrange method and maximize the volume

$$f = 2\pi r^2 h$$

under the constraint that $r^2 + h^2 = 1$.

Solution:

We have to extremize

$$f(x, y) = 2\pi x^2 y$$

under the constraint

$$g(x, y) = x^2 + y^2 = 1 .$$

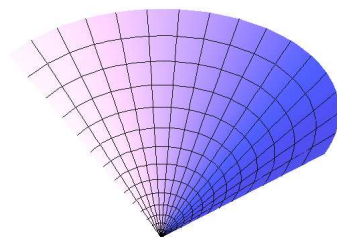
The Lagrange equations are

$$\begin{aligned} 4\pi r h &= \lambda 2r \\ 2\pi r^2 &= \lambda 2h \\ r^2 + h^2 &= 1 \end{aligned}$$

Divide the first by the second to get $2h/r = r/h$ or $2h^2 = r^2$ which gives $3h^2 = 1$. or

$$\boxed{h = 1/\sqrt{3}} \text{ and } \boxed{r = \sqrt{2}/\sqrt{3}} .$$

Problem 6) (10 points)



a) (6 points) Find the surface area of the surface

$$r(u, v) = [v^2 \cos(u), v^2 \sin(u), v^2]^T, 0 \leq u \leq \pi, 0 \leq v \leq 1 .$$

b) (4 points) Find the arc length of the boundary curve $\vec{r}(u, 1)$ where $0 \leq u \leq \pi$.

Solution:

a) $\vec{r}_u = [-v^2 \sin(u), v^2 \cos(u), 0]^T$.

$\vec{r}_v = [2v \cos(u), 2v \sin(u), 2v]^T$.

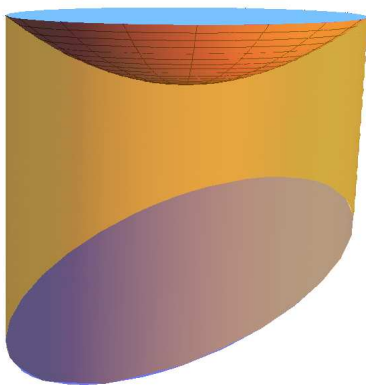
The cross product is $2v^3[\cos(u), \sin(u), -1]^T$ which has length $\sqrt{22}v^3$. Integrating this over the parameter domain gives

$$\int_0^\pi \int_0^1 v^3 \sqrt{22} \, dv du = \frac{v^4}{4} \Big|_0^1 \sqrt{22} = \pi \sqrt{2}/2 .$$

This is also $\boxed{\pi/\sqrt{2}}$.

b) The curve $\vec{r}(u) = [\cos(u), \sin(u), 0]^T$ has velocity $\vec{r}'(u) = [-\sin(u), \cos(u), 0]^T$ which has length 1. The arc length is $\boxed{\pi}$.

Problem 7) (10 points)



Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 + 4$.

Solution:

The best setup is in **cylindrical coordinates**, where $x - y = r \cos(\theta) - r \sin(\theta)$ and $x^2 + y^2 + 4 = r^2 + 4$:

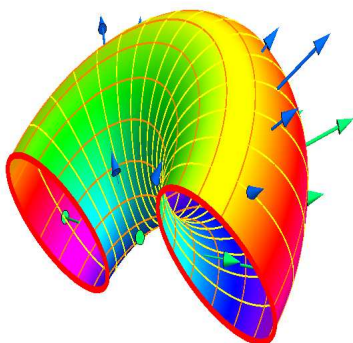
$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r \cos(\theta) - r \sin(\theta)}^{r^2 + 4} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} (r^2 + 4 - r \cos(\theta) + r \sin(\theta)) r \, dr \, d\theta .$$

Since integrating $\sin(\theta)$ and $\cos(\theta)$ from 0 to 2π is zero, we only have

$$\int_0^{2\pi} \int_0^{\sqrt{2}} (r^3 + 4r) \, dr \, d\theta = 2\pi (r^4/4 + 4r^2/2)|_0^{\sqrt{2}} = 2\pi \cdot 5 .$$

The answer is 10π .

Problem 8) (10 points)



Find the flux of the curl of the vector field

$$\vec{F}(x, y, z) = [x, y, z + \sin(\sin(y^2))]^T$$

through the torus

$$\vec{r}(s, t) = [(2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s)]^T$$

with $0 \leq t \leq \pi$ and $0 \leq s < 2\pi$.

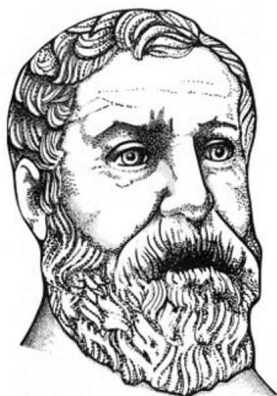
Solution:

This is a problem for Stokes theorem. We can find the boundary curves by setting $t = 0$ or $t = \pi$. The two curves are

$$C_1 : r_1(s) = [(2 + \cos(s)), 0, \sin(s)]^T, C_2 : r_2(s) = [-2 - \cos(s), 0, \sin(s)]$$

both parametrized for $0 \leq s \leq 2\pi$. The flux is the sum of the line integral of $\vec{F}$ along these two curves: $\int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} [2 + \cos(s), 0, \sin(s)]^T \cdot [-\sin(s), 0, \cos(s)]^T ds + \int_0^{2\pi} [-2 - \cos(s), 0, \sin(s)]^T \cdot [\sin(s), 0, \cos(s)]^T ds = 0 + 0 = 0$. The answer is 0.

Problem 9) (10 points)



Heron's formula for the area A of a triangle of side length $x, y, 1$ satisfies $16A^2 = f(x, y)$, where

$$f(x, y) = -1 + 2x^2 - x^4 + 2y^2 + 2x^2y^2 - y^4.$$

Classify all the critical points of f . Is there a global maximum of f and so for the area?

Remark not to worry about: The formula follows directly from Heron's formula $s = (a + b + 1)/2$; $A = \sqrt{s(s-a)(s-b)(s-1)}$.

Solution:

The critical points satisfy the equations $\nabla f(x, y) = [0, 0]^T$ which is

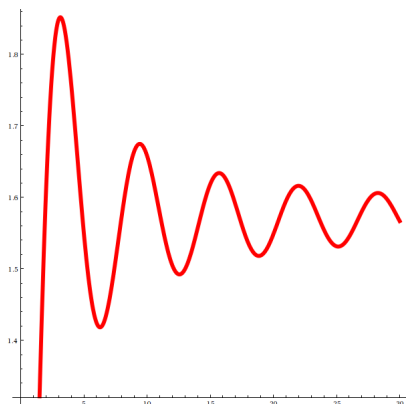
$$\begin{aligned} 4x(1 - x^2 + y^2) &= 0 \\ 4y(1 + x^2 - y^2) &= 0. \end{aligned}$$

To satisfy this there are four possibilities: either $x = 0, y = 0$ or $x = 0, 1 + x^2 - y^2 = 0$ or $1 - x^2 + y^2 = 0, y = 0$ or $1 - x^2 + y^2 = 0, 1 + x^2 - y^2 = 0$. This is $(0, 0), (0, \pm 1), (\pm 1, 0)$. We have $D(x, y) = 8xy$ and $f_{xx} = 4(1 - 3x^2 + y^2)$.

x	y	D	f_{xx}	classification	f
-1	0	-64	-8	saddle	0
0	-1	-64	8	saddle	0
0	0	16	4	minimum	-1
0	1	-64	8	saddle	0
1	0	-64	-8	saddle	0

There is no global maximum because $f(x, x) = 4x^2 - 1$ grows to infinity. There is no limit on areas of triangles we can build. $f(x, x)/16$ is the square of the area of an isoscele triangle with side lengths 1, x , x .

Problem 10) (10 points)



The anti derivative of the **sinc** function

$$\frac{\sin(x)}{x}$$

is called the **sine integral** $\text{Si}(x)$. It can not be expressed in terms of known functions. Still we can compute the following double integral

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} dy dx.$$

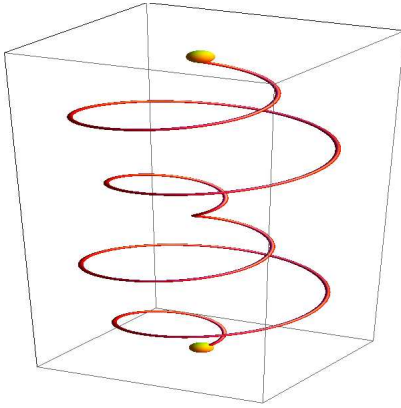
Solution:

Change the order of integration

$$\int_0^\pi \int_0^y \frac{\sin(y)}{y} dx dy = \int_0^\pi \sin(y) dy = -\cos(y)|_0^\pi = 2.$$

The answer is $\boxed{2}$.

Problem 11) (10 points)



Find the line integral of the vector field

$$\vec{F}(x, y, z) = [-x^{10}, \sin(y), z^3]^T$$

along the curve $\vec{r}(t) = [\sin(t) \cos(5t), \sin(t) \sin(5t), t]^T$ where $0 \leq t \leq 2\pi$.

Solution:

The vector field is a gradient field $\vec{F} = \nabla f$ with

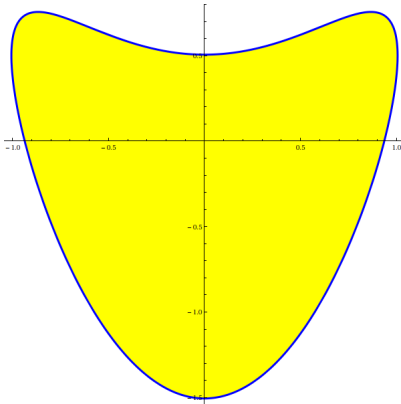
$$f(x, y) = -x^{11}/11 - \cos(y) + z^4/4 .$$

The curve end points are $f(r(2\pi)) = f(0, 0, 2\pi)$ and $f(r(0)) = f((0, 0, 0))$. The fundamental theorem of line integrals assures that

$$\int_C \vec{F} \, d\vec{r} = (2\pi)^4/4 - 0$$

which is $\boxed{4\pi^4}$.

Problem 12) (10 points)



Find the area of the region enclosed by the curve

$$\vec{r}(t) = [\cos(t), \sin(t) + \cos(2t)/2]^T ,$$

where $0 \leq t < 2\pi$.

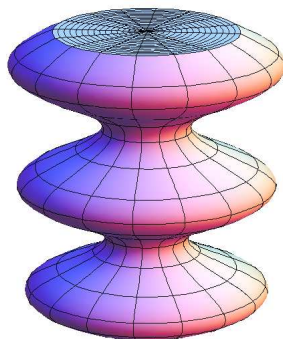
Solution:

Use the vector field $\vec{F}(x, y) = [0, x]^T$ which has constant curl 1. We can compute the area by computing the line integral of $\vec{F}$ along the boundary. This is

$$\int_0^{2\pi} [0, \cos(t)]^T \cdot [-\sin(t), \cos(t) - \sin(2t)]^T dt = \int_0^{2\pi} \cos^2(t) - 2\cos^2(t)\sin(t) dt = \int_0^{2\pi} \cos^2(t) dt = \pi.$$

The answer is $\boxed{\pi}$.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x, y, z) = [x^3/3, y^3/3, \sin(xy^5)]^T$$

through the boundary surface of the solid bound by the surface of revolution $\vec{r}(t, z) = [(2 + \sin(z)) \cos(t), (2 + \sin(z)) \sin(t), z]^T$ and the planes $z = 0, z = 3$. The surface is oriented so that the normal vector points outwards.

Solution:

We use the divergence theorem. Since $\text{div}(\vec{F}) = x^2 + y^2 = r^2$, we have to integrate this over the solid

$$\int_0^{2\pi} \int_0^3 \int_0^{2+\sin(z)} r^2 \cdot r dr dz d\theta = (2\pi) \int_0^3 \frac{r^4}{4} \Big|_0^{2+\sin(z)} dz = \frac{\pi}{2} \int_0^3 (2 + \sin(z))^4 dz.$$

It was sufficient to reach this integral $\boxed{\frac{\pi}{2} \int_0^3 (2 + \sin(z))^4 dz}$ to get full credit. The actual answer is $(\pi/64)(11756 - 3648 \cos(3) + 64 \cos(9) - 600 \sin(6) + 3 \sin(12))$.