7/24/2014 SECOND HOURLY PRACTICE VI Maths 21a, O.Knill, Summer 2014

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The function $f(x, y) = x + y$ has no critical points at all.

Solution:
Take $f(x, y) = x + y$ for example.

2) T F The gradient of f at a point (x_0, y_0, z_0) is orthogonal to the level surface of f which contains (x_0, y_0, z_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

3) T F If $(2, 3)$ is a local maximum for the function f with discriminant $D > 0$, then $f_{xx}(2, 3) < 0$.

Solution:
We can not have $f_{xx}(2, 3) > 0$. If $f_{xx}(2, 3) = 0$, then $D \leq 0$.

4) T F If f satisfies the partial differential equation $f_x + f_y = 0$ everywhere, then the discriminant D is zero at every critical point.

Solution:
Because $f_{xy} = f_{xx}$ and $f_{yx} = f_{yy}$, we have $D = f_{xx} f_{yy} - f_{xy}^2 = 0$.

5) T F If $f(x, y)$ has a saddle point at $(0, 0)$, then $-f(x, y)$ has a saddle point at $(0, 0)$.

Solution:
The same critical point is also a saddle point, if we turn it upside down.

6) T F The value of the function $f(x, y) = xy$ at $(x, y) = (3.1, 5.2)$ can be estimated as $15 + 0.5 + 0.6$.

Solution:
Use the formula for $L(x, y)$.

7) [✓ T] F — The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

Solution:
Unlike in the practice exam, this is now correct.

8) T [✓ F] — The gradient of f at a point (x_0, y_0) is parallel to the level curve of f which contains (x_0, y_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

9) T [✓ F] — If $(1, 1)$ is a critical point of $f(x, y)$ and is not a critical point of $g(x, y)$ then $(1, 1)$ can not be a critical point of f under the constraint g.

Solution:
It not only can, it always is. $\nabla f = \lambda \nabla g$ is solved with $\lambda = 0$.

10) [✓ T] F — If an airship always moves in the direction opposite to the gradient of the pressure, then the pressure momentarily decreases.

Solution:
Use the chain rule.

11) T [✓ F] — At points, where the velocity of a curve is zero, the curvature is zero.

Solution:
The curvature is not defined at such points.

12) T [✓ F] If D is the discriminant at a critical point and $D < 0$ then $f_{xy} \leq 0$.

Solution:
Take $f(x, y) = -xy$.

13) T [✓ F] If $(0, 0)$ is a critical point of $f(x, y)$ and $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) > 0$, then $(0, 0)$ can not be a local minimum.

Solution:
Take $x^2 + y^2 - 100xy$ for example

14) T [✓ F] The tangent plane to the graph $f(x, y) = z$ at a point (x_0, y_0) is parallel to $\langle 0, 0, 1\rangle$ if (x_0, y_0) is a critical point.

Solution:
It is perpendicular to the surface, not prallel .

15) T [✓ F] The directional derivative $D_{\vec{v}} f$ is a vector perpendicular to $\vec{v}$.

Solution:
The directional derivative is a scalar, not a vector.

16) [✓ T] F The integral $\int_0^1 \int_0^1 x^2 + y^2 \, dxdy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.

Solution:
In general $\int \int_R f(x, y) \, dydx$ is the volume under the graph of f

17) [✓ T] F The directional derivative $D_v f(1, 1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1, 1)$.

Solution:
The level curve is perpendicular to the gradient.

18) T [✓ F] If (a, b) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a, b).

Solution:
False, by changing g to $-g$, we can change the Lagrange multiplier, but the discriminant stays the same.

19) [✓ T] F The surface area of a surface is independent of the parametrization

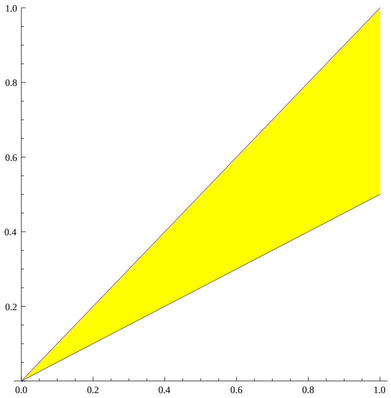
Solution:
Yes, this is a basic property like arc length for curves.

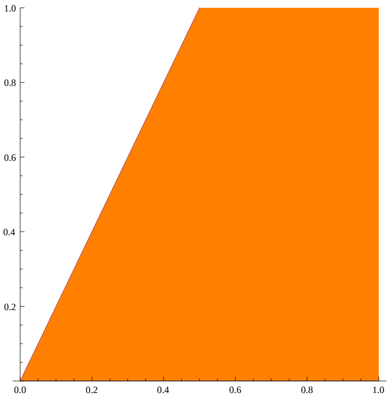
20) T [✓ F] The double integral $\int_0^1 \int_0^1 xy \;\; dxdy$ is in polar coordinates the integral $\int_0^1 \int_0^1 r^2 \cos(\theta) \sin(\theta) \; r \; drd\theta$.

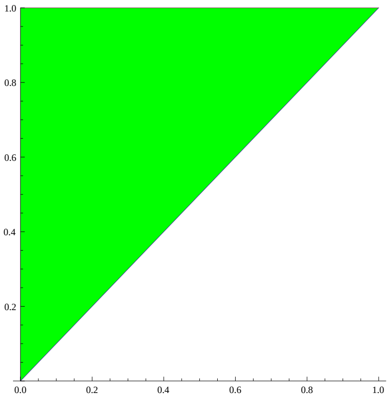
Solution:
We also have to transform the parameter range.

Problem 2) (10 points)

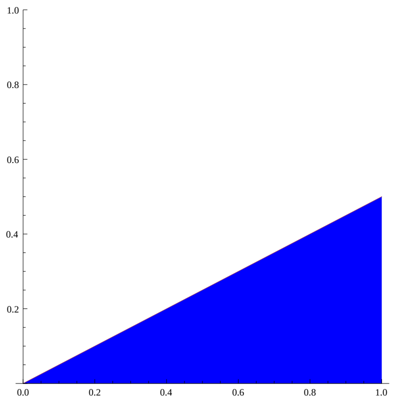
Match the regions with the corresponding double integrals

a


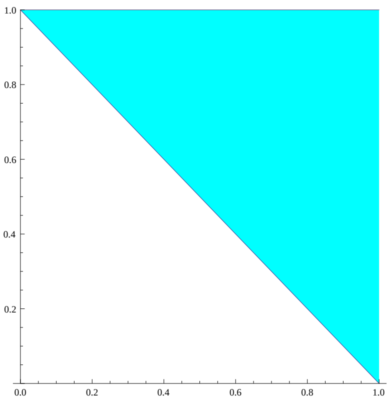
b


c


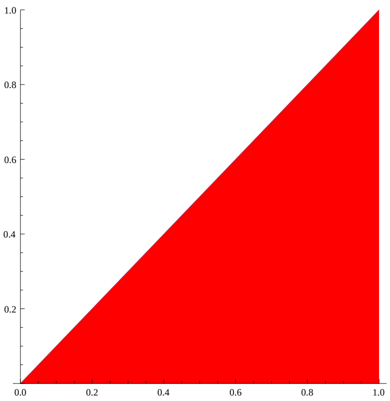
d



e



f



Enter a,b,c,d,e or f	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x/2}^{x} f(x,y)\ dydx$
	$\int_0^1 \int_0^{y} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{x/2} f(x,y)\ dydx$
	$\int_0^1 \int_{y/2}^{1} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{x} f(x,y)\ dydx$
	$\int_0^1 \int_{1-x}^{1} f(x,y)\ dydx$

Solution:
a,c,d,b,f,e.

Problem 3) (10 points)

a) (4 points) Find all the critical points of the function $f(x,y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 \ .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Solution:
a) $\nabla f(x,y) = \langle y, x \rangle = \langle 0, 0 \rangle$ implies $x = y = 0$. The only critical point in the interior is $\boxed{(0,0)}$. The discriminant is $D = -1^2 = -1$. The point is a $\boxed{\text{saddle point}}$.
b) With $g(x,y) = x^2 + y^2/4$, we have the Lagrange equations

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 2y/4 \\ x^2 + y^2/4 &= 1 \end{aligned}$$

Dividing the first equation by the second to get $y/x = 4x/y$ which means $y^2 = 4x^2$ or $y = \pm 2x$. The third equation gives $2x^2 = 1$ or $x = \pm 1/\sqrt{2}$. The third equation gives $y = 2\sqrt{1-x^2} = \pm\sqrt{2}$. The critical points are

$$\{(1/\sqrt{2}), \sqrt{2}\}, \{(-1/\sqrt{2}), \sqrt{2}\}, \{(1/\sqrt{2}), -\sqrt{2}\}, \{-(1/\sqrt{2}), -\sqrt{2}\} .$$

The value of the function at these points are $1, -1, -1, 1$. The first and last are maxima, the second and third are minima.
c) There is no global maximum, nor a global minimum in the interior of the disc because there is no local maximum in the interior of the disc. The global maxima as well as the minima are on the boundary.

Problem 4) (10 points)

Oliver rides his bike along streets in the Massachusetts. Since the streets can be quite bumpy, he tries to avoid critical points which are maxima (bumps) or minima (potholes) but aims to drive over saddle points (mountain passes). Assume the street is the graph of the function

$$f(x,y) = \frac{x^4}{4} + \frac{y^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} .$$

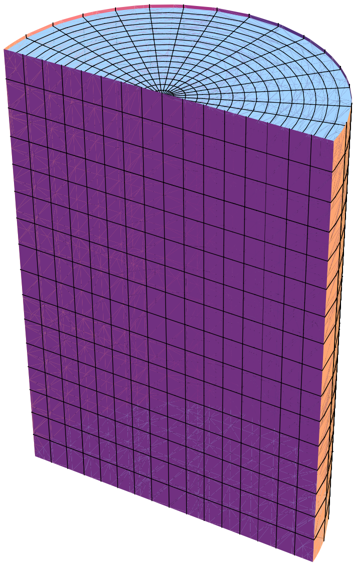
List all critical points and classify them as local maxima, local minima and saddle points.

Solution:
$\nabla f(x,y) = x^3 - x, y^3 + y = (0,0)$ so that the critical points are $(-1,0), (0,0), (1,0)$. We have $D = (3x^2-1)(3y^2+1)$ and $f_{xx} = -1+3x^2$.

Point	D	f_{xx}	type
$(-1,0)$	$D=2$	2	min
$(0,0)$	$D=-1$	-1	saddle
$(1,0)$	$D=2$	2	min

Problem 5) (10 points)

Its amazing how many new energy drinks pop up every year and each tries to be original. Beside the concept like ”water from Norway” or ”vitamin drinks” or ”taurin”, ”coffeine bomb”, it needs also a catchy name ”pink bull”, ”muscle milk” etc, and a fancy shape. Summer school is always a good time to come up with business ideas. We launch in this midterm an energy drink called ”sweet tooth” which contains a insanely strong ”caffeine, taurin, ginseng, isotonic vitamin” combination. But what really makes it stand out from the crowd is its shape: its half a cylinder of radius x and height y. Its only for tough guys or girls although: one first has to bite off a corner of the can with the teeth, in order to drink it ...



For a fixed volume $y\pi x^2/2 = \pi/2$, we want to minimize the material cost $\pi xy + \pi x^2 + 2xy$. In other words, we want to minimize the function

$$f(x,y) = (2+\pi)xy + \pi x^2$$

under the constraint

$$g(x,y) = yx^2 = 1 \ .$$

Solve it with Lagrange method!

Solution:
The Lagrange equations are

$$\begin{aligned} (2+\pi)y + 2\pi x &= 2xy\lambda && (1) \\ (2+\pi)x &= \lambda x^2 && (2) \\ yx^2 &= 1 && (3) \end{aligned}$$

From the first two equations (divide the first by the second using that $x = 0$ is not compatible with the third equation, gives $y = 2\pi x/(2+\pi)$. Plugging this into the third equation gives the relation $x = [(2+\pi)/(2\pi)]^{1/3}$ and $y = [2\pi/(2+\pi)]^{2/3}$.

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle t^3 + 3, 2 + \cos(t^3), 3 + \sin(t^3) \rangle$$

from $t = 0$ to $t = \pi$.

Solution:
The velocity is $\vec{r}\,'(t) = 3t^2 \langle 1, -\sin(t), \cos(t) \rangle$. The speed is $3t^2\sqrt{2}$. The integral

$$L = \int_0^\pi 3t^2\sqrt{2} = \pi^3\sqrt{2}.$$

Problem 7) (10 points)

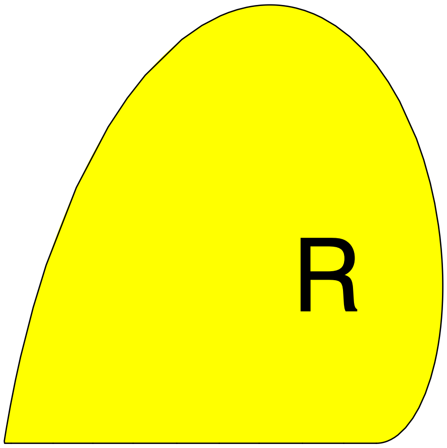
Estimate $f(x,y) = \sqrt{x^3 + y^3}$ for $x = 1.001, y = 2.01$ by linear approximation.

Solution:
$\nabla f(x,y) = \langle \frac{3x^2}{2\sqrt{x^3+y^3}}, \frac{3x^y}{2\sqrt{x^3+y^3}} \rangle$. At the point $(1,2)$ where $f(1,2) = 3$, we have and $\nabla f(1,2) = \langle 1/2, 2 \rangle$. We have $f(1.001, 2.01) \sim 3 + (1/2) \cdot 0.001 + 2 \cdot 0.01 = 3.0205$.

Problem 8) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}(\pi - \sqrt{x^2+y^2})} \; dx \, dy \; .$$



Solution:
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = dr d\theta$ does not work. The integral $\int_0^\pi \int_0^\theta \frac{\cos(r)}{r(\pi-r)} r dr d\theta$ can not be solved. We have to change the order of integration:

$$\int_0^\pi \int_r^\pi \frac{\cos(r)}{r(\pi - r)} r \; d\theta \; dr$$

Evaluating the inner integral gives $\int_0^\pi \cos(r) \; dr = \boxed{0}$.

Problem 9) (10 points)

a) (5 points) Find the tangent plane to the surface $xy + x^2y + zx^2 = 20$ at the point $(2, 2, 2)$.
b) (5 points) Find the tangent line to the curve $x^2y^2 - xy + 3x = 3$ at the point $(1, 1)$.

Solution:
a) The gradient is $\langle y+2xy+2xz, x+x^2, x^2\rangle$. At the point $(2,2,2)$ the gradient is $\langle 18, 6, 4$. The plane has the form $18x + 6y + 4z = d$. By pluggin in the point $(2,2,2)$ we get $18x + 6y + 4z = 56$.
b) The gradient is $\langle 2xy^2 - y + 3, 2x^2y\rangle$. At the point $(1,1)$ it is $4, 1$. The line is $4x + y = d$. By plugging in the point $(1,1)$ we get $4x + y = 5$.

Problem 10) (10 points)

Find the tangent plane to the surface

$$\sin(x+y) - \cos(z-x) + \sin(y) = -1$$

at the point $(0, \pi, 0)$.

Solution:
We compute the gradient $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$ with $f_x = \cos(x+y) + \sin(x-z)$, $f_y = \cos(y) + \cos(x+y)$ and $f_z = -\sin(x-z)$. At the point $(0,\pi,0)$ we have $\nabla f(0,\pi,0) = \langle -1, -2, 0\rangle = \langle a, b, c\rangle$. The equation for the tangent plane is $ax + by + cz = d$ where d is obtained by plugging in $(x,y,z) = (0,\pi,0)$. The equation is $-x - 2y = -2\pi$ which can also be written as $\boxed{x + 2y = 2\pi}$.