7/24/2014 SECOND HOURLY PRACTICE IV Maths 21a, O.Knill, Summer 2014

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The partial differential equation $u_t = u_{xx}$ is called the heat equation.

Solution:
This was a knowledge question.

2) T F Every function $f(x, y)$ in 2 variables has either a local maximum or a local minimum or a saddle point.

Solution:
We can have no critical point at all like $f(x, y) = x + y$.

3) T F If $\iint_R f(x, y)\ dxdy = 0$, then the function $f(x, y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.

Solution:
If $f = xy$ and R is the disc, then the integral is zero but f is nonzero.

4) T F If f has a local maximum at $(1, 0)$ and $g(x, y) = x^2 + y^2 = 1$ is a constraint then the Lagrange equations $\nabla f = \lambda \nabla g, g = 1$ are satisfied.

Solution:
Just take $\lambda = 0$. The point is on the constraining curve.

5) T F The directional derivative is always smaller than the length of the gradient.

Solution:
It can also be equal to the length.

6) T F The linearization of a function $f(x, y)$ at $(0, 0)$ is a function of the form $L(x, y) = ax + by + c$.

Solution:
By definition

7) T F The surface area of a surface $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ defined on $(u,v) \in R$ is smaller or equal to than $\int\int_R |\vec{r}_u| \cdot |\vec{r}_v| \; dudv$.

Solution:
We have $|\vec{r}_u \times \vec{r}_v| \leq |r_u| \cdot |r_v|$.

8) T F A rectangle is both type I and type II.

Solution:
By definition of type I and type II.

9) T F If the function $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$ is negative everywhere, then every critical point of f is a saddle point.

Solution:
This is a consequence of the second derivative theorem.

10) T F If two functions f and g have the same critical points, then $f = \lambda g$.

Solution:
$f(x,y) = x^2 + y^2$ and $g(x,y) = x^2 - y^2$ are already a counter example.

11) T F If (x,y) is not a critical point, then the directional derivative $D_{\vec{v}}f(x,y)$ can take both positive and negative values for different choices of $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

12) T F

Using linearization of $f(x,y) = x/y$ we can estimate $1.01/1.1 = f(1.01, 1.1) \sim 1 + 0.01 - 0.1 = 0.91$.

Solution:
$L(x,y) = 1 + 1 \cdot 0.01 - 1 \cdot 0.1$.

13) T F

If $(1,1)$ is a local maximum, then $D = f_{xx}^2 f_{yy}^2 - f_{xy}^2 \geq 0$.

Solution:
If D would be the discriminant were negative then it would be a saddle point and therefore not a local minimum. The function is however not the discriminant. The function $f(x,y) = a(x^2+y^2) + 2xy$ has $D = 16a^4 - 4 > 0$ at the origin if $a > 1/\sqrt{2}$ and the usual discriminant is $4a^2 - 4$ is negative if $a < 1$.

14) T F

$\int_0^2 \int_0^3 f(x,y)\, dxdy = \int_0^2 \int_0^3 f(x,y)\, dydx$ for any continuous function $f(x,y)$.

Solution:
We also have to switch the integration bounds.

15) T F

If $\vec{r}(t)$ is a curve for which the speed is 1 at all times and f is a function, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

Solution:
Yes, this is the chain rule.

16) T F

Assume f is zero on the boundary of the unit square $R = \{0 \leq x \leq 1, 0 \leq y \leq 1\}$, then $\int_0^1 \int_0^1 f_{xy}(x,y)\, dydx = 0$.

Solution:
It is a consequence of the fundamental theorem of calculus that we can replace for any x the integral by its boundary points.

17) T F

If $f_{yy}(x,y) < 0$ for all x, y, then f can not have any local minimum.

Solution:
We would have $f_{yy} > 0$ at a local minimum.

18) | T | F |

The double integral $\int_{-1}^{1}\int_{-1}^{1} x^2 - y^2 \, dxdy$ is the volume of the solid below the graph of $f(x,y) = x^2 - y^2$ and above the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ in the xy-plane.

Solution:
It is a signed volume. There can be part below.

19) | T | F |

For any function f and any unit vector $\vec{v}$ one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

Solution:
Write down the definition. The sum is $\nabla f \cdot (\vec{v} - \vec{v}) = 0$.

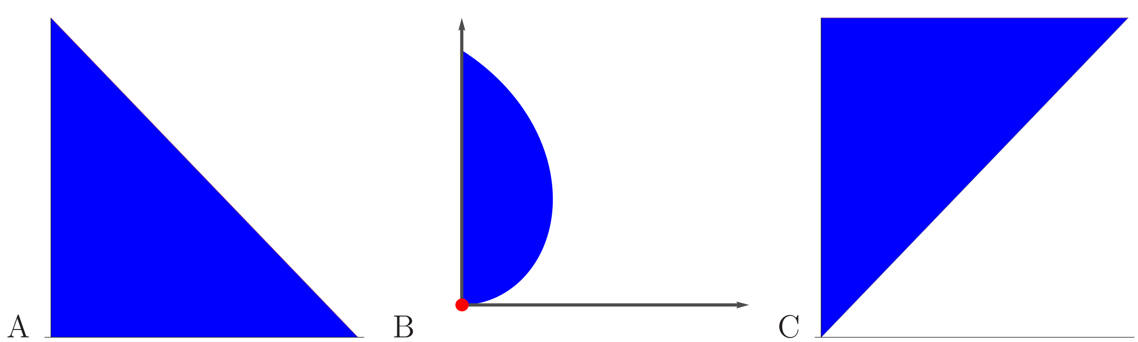
20) | T | F |

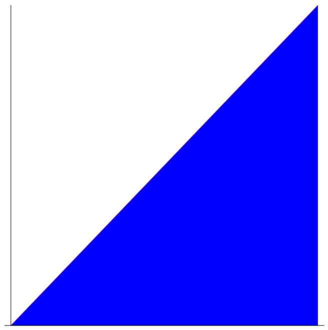
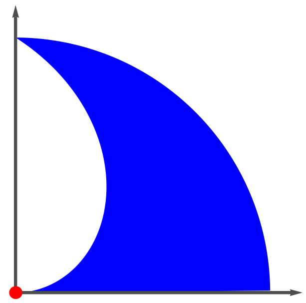
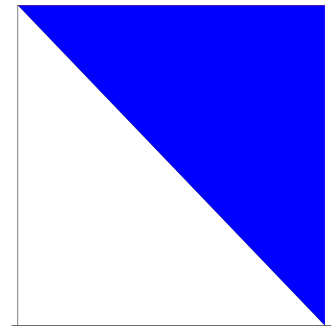
The surfaces $x^2 + y^2 + z^2 = 2$ and $x^2 - y^2 + z^2 = 2$ have the same tangent plane at $(1, 0, 1)$.

Solution:
They have the same gradient at $(0, 0, 0)$.

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the regions with the double integrals. If none applies, put O.



D  E  F 

Enter A-F	Integral of Function $f(x,y)$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{y} f(x,y)\ dxdy$
	$\int_0^{\pi/2}\int_0^{x} f(x,y)\ dydx$

Enter A-F	Integral of Function $f(x,y)$
	$\int_0^{\pi/2}\int_{\theta}^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/2-y}^{\pi/2} f(x,y)\ dxdy$
	$\int_0^{\pi/2}\int_0^{\pi/2-x} f(x,y)\ dydx$

b) (4 points) Assume $\vec{v} = \nabla f(1,1) = \langle 3/5, 4/5\rangle$ and $\vec{r}(t) = \langle 1+3t, 1+4t\rangle$ and that $L(x,y)$ is the linearization of f at $(1,1)$ and $f(1,1) = 2$. Finally, denote by $ax + by = d$ the tangent line of f at $(1,1)$.

Here is a choice of 5 answers. 4 of them apply above, fill them in. If none should apply, you would enter O.

Expression	Fill in A-E
$D_{\vec{v}}f(1,1)$	
$d/dt f(\vec{r}(t))\vert_{t=0}$	
$L(2,1)$	
the constant d in the tangent line	

A	5
B	7/5
C	13/5
D	25
E	1

Solution:

Enter A-F	Integral of Function $f(x,y)$
B	$\int_0^{\pi/2} \int_0^{\theta} f(r,\theta) r \, dr d\theta$
C	$\int_0^{\pi/2} \int_0^{y} f(x,y) \, dx dy$
D	$\int_0^{\pi/2} \int_0^{x} f(x,y) \, dy dx$

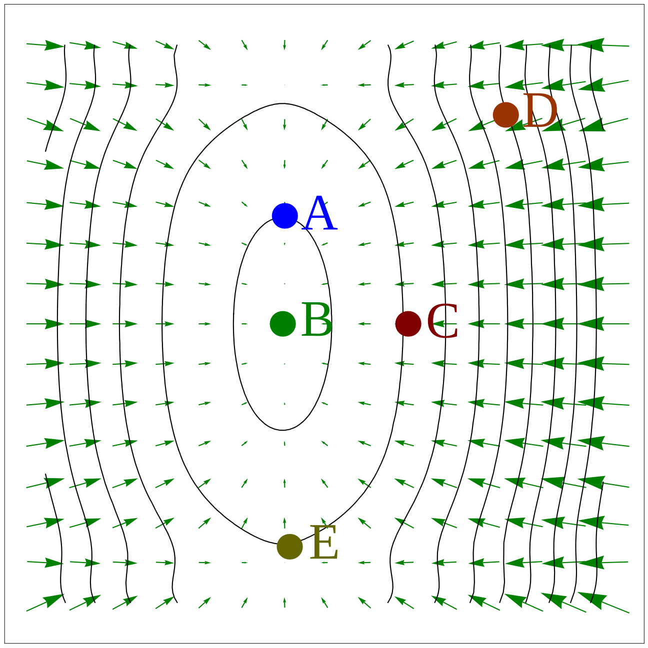
Enter A-F	Integral of Function $f(x,y)$
E	$\int_0^{\pi/2} \int_{\theta}^{\pi/2} f(r,\theta) r \, dr d\theta$
F	$\int_0^{\pi/2} \int_{1-y}^{\pi/2} f(x,y) \, dx dy$
A	$\int_0^{\pi/2} \int_0^{\pi/2--x} f(x,y) \, dy dx$

b) E, A, C, B.

Problem 3) (10 points) (no justifications are needed)

a) (5 points) A function $f(x,y)$ of two variables has level curves as shown in the picture. The arrows are the gradient.

Enter A-E or O if no match	Description
	a local maximum $f(x,y)$.
	a local minimum $f(x,y)$.
	a point, where $f_x = 0$ and $f_y < 0$
	a point, where $f_y = 0$ and $f_x < 0$
	the point among $A - E$, where $\|\|\nabla f\|\|$ is largest



b) (5 points) Assume $f(x,y) = x^2 - y^2 + 1, g(x,y) = x^2 + y^2 + 1$. Which of the following statements are true (T) or false (F):

Enter T/F	Statement
	$(0,0)$ is a critical point for f.
	$(0,0)$ is a critical point for g.
	$(0,0)$ is a critical point for f under the constraint $g = 1$.
	$(0,0)$ is a critical point for g under the constraint $f = 1$
	$(0,0)$ is a global minimum for g.

Solution:
a) $BOACD$.
b) $TTTTT$.

Problem 4) (10 points)

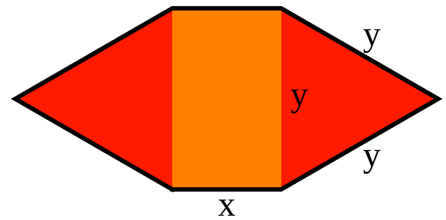
A hexagonal shape is made of two equilateral triangles of side length y and a rectangle of length x and width y. The area is

$$f(x,y) = xy + (\sqrt{3}/2)y^2$$

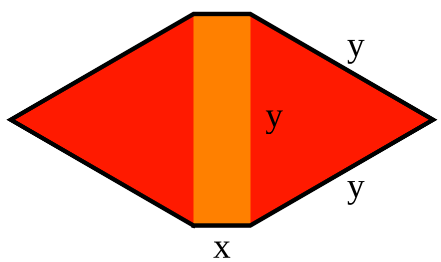
the circumference is

$$g(x,y) = 2x + 4y\ .$$

Assume the circumference is fixed so that $g(x,y) = 8$. Find the dimensions x,y for which the hexagon has maximal area. Use the method of Lagrange multipliers.



Solution:
The Lagrange equations $\nabla f = \langle y, x + \sqrt{3}y \rangle = \lambda \langle 2, 4 \rangle$, $x + 2y = 4$. Dividing the first equation by the second and plugging into the third equation The solution is $x = 4(5 - 2\sqrt{3})/13 = 0/472$ and $y = 4(4 + \sqrt{3})/13 = 1/763$.



Solution shape.

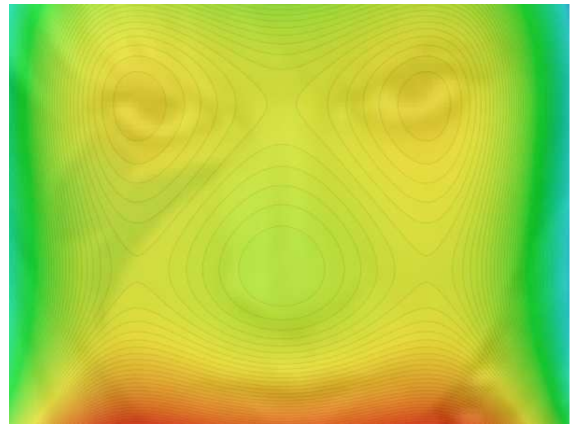
Problem 5) (10 points)

a) (8 points) The face of Lady Gaga - a women of extremes - is modeled by the **Gaga function**

$$f(x,y) = 2y^3 - 3y^2 + x^4 - 2x^2\ .$$

Classify all critical points of f. Which are maxima, minima or saddle points.

b) (2 points) Is there a global maximum for f or is there no maximal "Gaga"?

Solution:
The gradient is

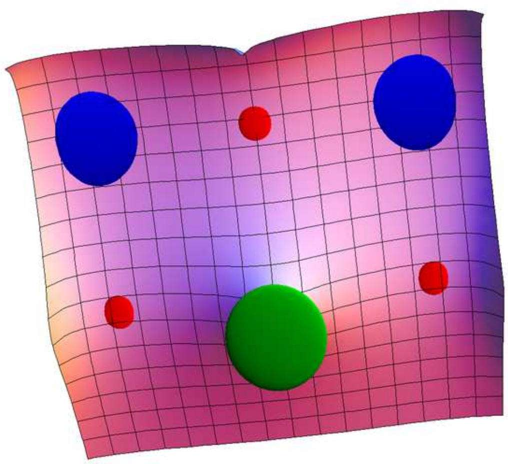
$$\nabla f(x,y) = \langle -4x + 4x^3, -6y + 6y^2 \rangle \ .$$

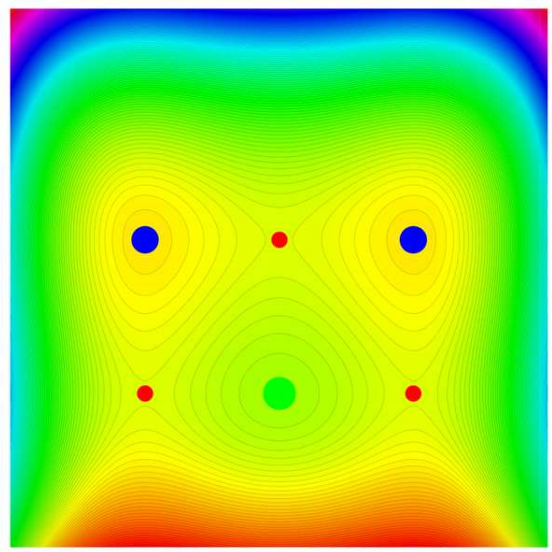
The Hessian is

$$H(x,y) = \begin{bmatrix} 12x^2 - 4 & 0 \\ 0 & 12y - 6 \end{bmatrix}$$

so that $D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y - 6)$ and $f_{xx} = 12x^2 - 4$. The solutions are

-1	0	-48	8	*saddle*	-1
-1	1	48	8	*minimum*	-2
0	0	24	-4	*maximum*	0
0	1	-24	-4	*saddle*	-1
1	0	-48	8	*saddle*	-1
1	1	48	8	*minimum*	-2





Problem 6) (10 points)

Find the surface area of the surface

$$\vec{r}(t,s) = \langle t^2 + 1, s^2 - 1, 1 + \sqrt{2}ts \rangle \ .$$

for which the parameters satisfy $t^2 + s^2 \le 9$.

Solution:
We have

$$\vec{r}_t(t,s) = \langle 2t, 0, \sqrt{2}s \rangle$$

and

$$\vec{r}_s(t,s) = \langle 0, 2s, \sqrt{2}t \rangle$$

so that

$$\vec{r}_t \times \vec{r}_s = \langle -\sqrt{8}s^2, -\sqrt{8}t^2, 4ts \rangle$$

and

$$|\vec{r}_t \times \vec{r}_s| = \sqrt{8}(s^2 + t^2)$$

To integrate over the parameter region, we use polar coordinates:

$$\int_0^{2\pi} \int_0^3 \sqrt{8} r^2 \; r \; dr d\theta \; = 81\sqrt{2}\pi \; .$$

Problem 7) (10 points)

a) (5 points) Find the linearization function $L(x,y)$ of $f(x,y) = \sqrt{x^3 y}$ at $(x,y) = (1,4)$.

b) (5 points) Estimate

$$\sqrt{0.999^3 \cdot 3.9}$$

by evaluating $L(0.999, 3.9)$.

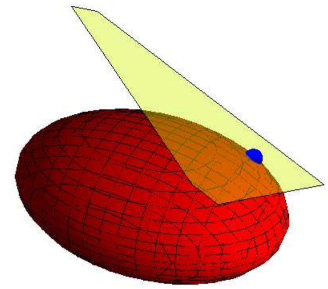
Solution:
The gradient at $(1,4)$ is $\langle 3, 1/4 \rangle$ and the function value is $f(1,4) = 2$.
a) $L(x,y) = 2 + 3(x_1) + \frac{1}{4}(y-4)$
b) $2 - 0.003 - 0.025 = \boxed{1.972}$.

Problem 8) (10 points)

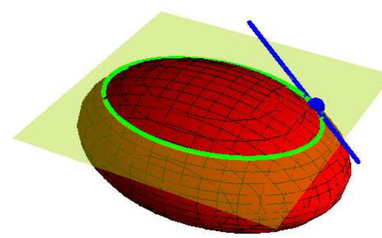
a) (5 points) Find an equation of the form $ax+by+cz = d$ which gives the tangent plane to the ellipsoid

$$4x^2 + 9y^2 + 16z^2 = 41$$

at the point $(2,1,1)$.



b) (5 points) If we intersect the ellipsoid with a plane $z = 1$, we obtain an ellipse $g(x,y) = 4x^2 + 9y^2 = 25$. Find an equation of the form $ax+by = d$ for the tangent line to that level curve $g(x,y) = 25$ at the point $(2,1)$.



Solution:
a) $\nabla f(x,y,z) = \langle 8x, 18y, 32z\rangle$ and $\nabla f(2,1,1) = \langle 16, 18, 32\rangle$. The tangent plane (plug in the point $(2,1,1)$ to get the constant d.) is

$$16x + 18y + 32z = 82 \ .$$

b) $\nabla g(x,y) = \langle 8x, 18y\rangle$ and $\nabla g(2,1) = \langle 16, 18\rangle$. The tangent line through the point $(2,1)$ is

$$16x + 18y = 50 \ .$$

Problem 9) (10 points)

Two underwater robots $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ and $\vec{R}(t) = \langle 1 + \sin(t), 1 - \cos(t) \rangle$ circle the just closed BP oil cap in the gulf of Mexico. The oil concentration is an unknown function $f(x, y)$ which is normalized that $f(1, 0) = 0$. The robots measure $D_{\vec{r}\,'(t)} f = 4$ and $D_{\vec{R}\,'(t)} f = 5$ at $t = 0$.

a) (5 points) Find the gradient $\nabla f(1, 0) = \langle a, b \rangle$ of f at $(1, 0)$.

b) (5 points) Estimate the oil concentration $f(1.1, 0.02)$ at the point $(1.1, 0.02)$.

Solution:
a) Set $\nabla f(1, 0) = \langle a, b \rangle$ so that we can find its components a, b. We know $\vec{r}\,'(0) = \langle 0, 1 \rangle$ and $\vec{R}\,'(0) = \langle 1, 0 \rangle$. The chain rule tells us that $D_{\langle 0,1 \rangle} f = b = 5$. Similarly $D_{\langle 1,0 \rangle} f = a = 4$. We know therefore $\nabla f(1, 0) = \langle 5, 4 \rangle$. The answer is $\boxed{\langle 5, 4 \rangle}$.
b) If we know the gradient and the function value $f(1, 0) = 0$, we can write down the linearization as

$$L(x, y) = f(1, 0) + a(x - 1) + b(y - 0) = 0 + 5(x - 1) + 4y \ .$$

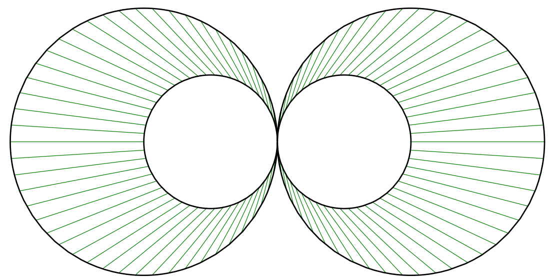
To estimate the oil concentration we evaluate L at the given point and get $L(1.1, 0.02)) = 0.5 + 0.08 = 0.58$. The answer is $\boxed{0.58}$.

Problem 10) (10 points)

Find the area of the region in the plane given in polar coordinates by

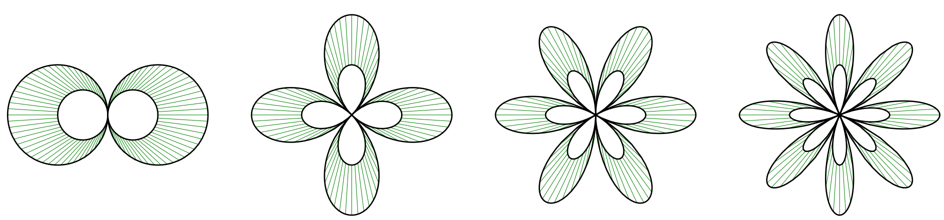
$$\{(r, \theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi \} \ .$$

The region is the shaded part in the figure.



Solution:
$\int_0^{2\pi} \int_{|\cos(\theta)|}^{|2\cos(\theta)|} r \; dr d\theta = \int_0^{2\pi} 4\cos^2(\theta)/2 - \cos^2(\theta)/2 \; d\theta = \boxed{3\pi/2}$. The result is the same for any region

$$\{(r,\theta) \mid |\cos(n\theta)| \leq r \leq 2|\cos(n\theta)|, 0 \leq \theta < 2\pi \} \; .$$



n=1 n=2 n=3 n=4