7/24/2014 SECOND HOURLY PRACTICE II Maths 21a, O.Knill, Summer 2014

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | The partial differential equation $u_{tt} = u_{xx}$ is an example of a heat equation.

Solution:
Knowledge question. It is the wave equation

2) | **T** | F | The level curve $f = 0$ of $f(x, y) = x^2 - y^2$ intersects with the unit circle in 4 points.

Solution:

3) | T | **F** | The critical points of D are called the discriminants of $f(x, y)$.

Solution:
This is nonsense. D is the discriminant of the critical point.

4) | **T** | F | The relation $f_{xyx}(x, y) = f_{yxx}(x, y)$ holds for all points (x, y) if f is the function $f(x, y) = y^9 \sin(\sin(\cos(\sin(yx^4))))$.

Solution:
This is a consequence of Clairot's theorem.

5) | **T** | F | The tangent plane to the surface $x^2 + y^2 + z^2 = 1$ at the point $(1, 0, 0)$ is $x = 1$.

Solution:
No need to compute the gradient. This is the unit sphere and at $(1, 0, 0)$ the plane $x = 1$ is tangent. Of course, no harm done by computing the gradient $(2, 0, 0)$ at $(1, 0, 0)$ and get $2x = d$ with finding d by plugging in the point $x = 1$ so that $2x = 2$.

6) T **F** Fubini's theorem implies that for any function $f(x,y)$ of two variables, we have $\int_0^1 \int_3^4 f(x,y)\, dxdy = \int_0^1 \int_3^4 f(x,y)\, dydx$.

Solution:
A counter example is $f(x,y) = y$.

7) T **F** If a function $f(x,y)$ has a saddle point at $(0,0)$ then $D_{\vec{v}}f(0,0)$ takes positive and negative values at $(0,0)$.

Solution:
It is constant zero.

8) **T** F The integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dydx$ is the area of the unit disc.

Solution:
Yes. This is the area in Cartesian coordinates.

9) T **F** There is a function $f(x,y)$ for which the linearization at $(0,0)$ is $L(x,y) = 1 + x^2 + 2y$.

Solution:
The linearization is a linear function and not quadratic.

10) T **F** The height of the Mont Blanc is $f(x,y) = 4810 - 2x^2 - y^2$. At height 4810 meters, the directional derivative in the direction $(1,0)$ is negative.

Solution:
It is a local maximum so that the directional derivatives are zero

11) **T** F If $\vec{r}(t)$ is a curve on the hyperbolic paraboloid $g(x,y,z) = x^2 - y^2 + z = 10$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Use the chain rule and the fact that $g(\frac{r}{(}t))$ is constant so that $d/dtg(\frac{r}{(}t))$ is zero.

12) T F

If the length $|\nabla f(0,0)|$ of the gradient is equal to 1, then there is a direction for which the slope of the graph of f at $(0,0)$ is 1.

Solution:
It is the direction of the gradient

13) T F

If $f(x,y)$ has a local minimum at $(0,0)$, then the discriminant satisfies $D > 0$ and furthermore $f_{xx} < 0$.

Solution:
Not necessarily. An counter example is $f(x,y) = x^4 + y^4$.

14) T F

$\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_0^y f(x,y)\, dydx$.

Solution:
The integrals can not be switched.

15) T F

The area of the half disk of radius L in the half plane above the x-axes is $\int_0^\pi \int_0^L r\, drd\theta$.

Solution:
Yes this is a polar integral.

16) T F

If $f_{xy}(0,0) = 0$, then f has a local maximum or a local minimum at $(0,0)$.

Solution:
The example $x^2 - y^2$ is already a counter example.

17) T F

The gradient vector $\nabla f(x,y)$ is a vector in space perpendicular to the graph of $f(x,y)$.

Solution:
Big misconception. This gradient vector is a vector in the plane, not in space.

18) T **F** — The surface area of the Gabriel trumpet $r = 1/z, 1 \leq r < \infty$ is finite because the volume of the inside is finite.

Solution:
We have seen in class that it is infinite.

19) **T** F — If the directional derivative at a point $(0,0)$ is known in the direction $\langle 1,1\rangle/\sqrt{2}$ and $\langle 1,2\rangle/\sqrt{5}$ then the gradient $\nabla f(0,0)$ is known.

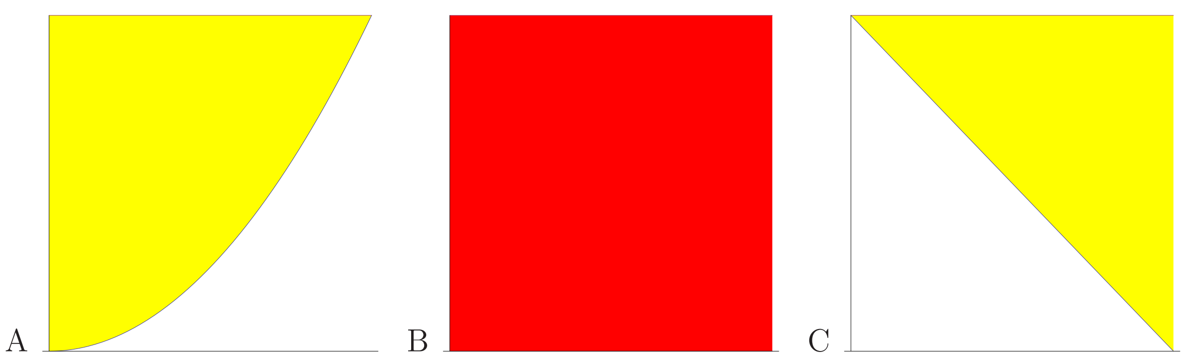
Solution:
We can compute the gradient $\langle a,b\rangle$ by solving a small system of equations.

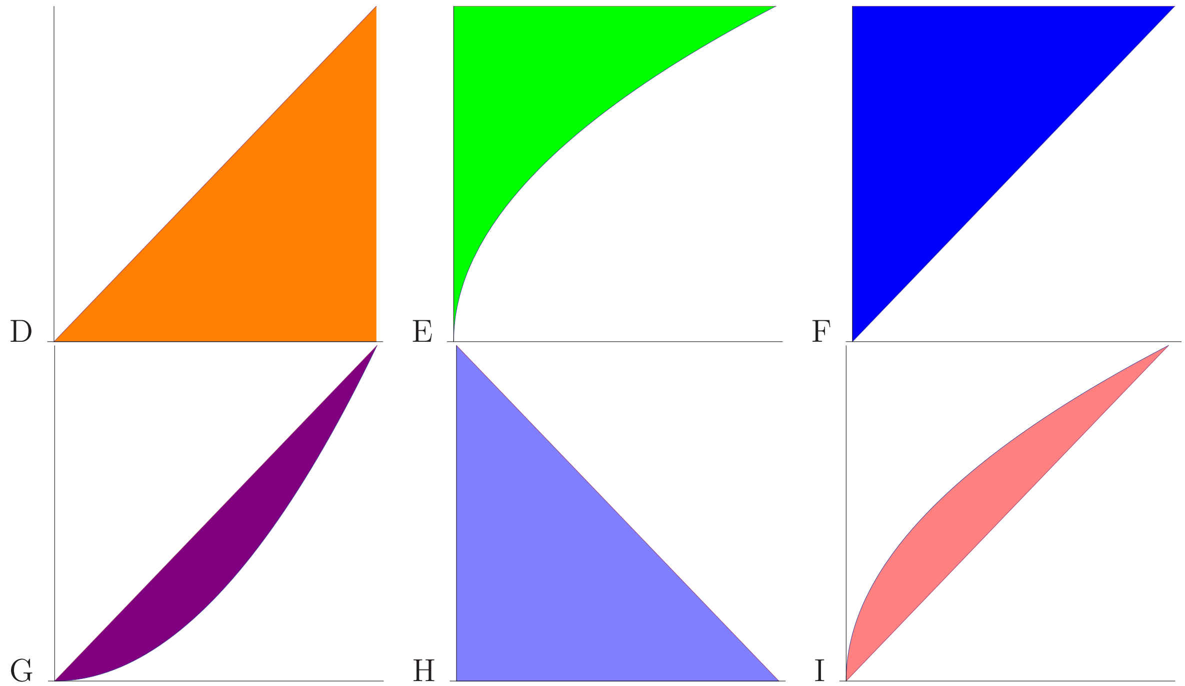
20) **T** F — If $f_x(x,y) = f_y(x,y)$ then every critical point (x_0,y_0) of f has zero discriminant $D = 0$.

Solution:
As discussed in the review, you can directly check that $D = 0$ because $f_{xy} = f_{xx}$ and $f_{yx} = f_{yy}$.

Problem 2) (10 points) No justifications are needed

(10 points) Match the regions with the double integrals.





A-I	Integral
	$\int_0^2 \int_{x^2/2}^2 f(x,y)\ dydx$
	$\int_0^2 \int_0^x f(x,y)\ dydx$
	$\int_0^2 \int_0^2 f(x,y)\ dydx$

A-I	Integral
	$\int_0^2 \int_0^{y^2/2} f(x,y)\ dxdy$
	$\int_0^2 \int_0^y f(x,y)\ dxdy$
	$\int_0^2 \int_{y^2/2}^y f(x,y)\ dxdy$

A-I	Integral
	$\int_0^2 \int_{x^2/2}^x f(x,y)\ dydx$
	$\int_0^2 \int_0^{2-x} f(x,y)\ dydx$
	$\int_0^2 \int_{2-x}^2 f(x,y)\ dydx$

Solution:

A	E	G
D	F	H
B	I	C

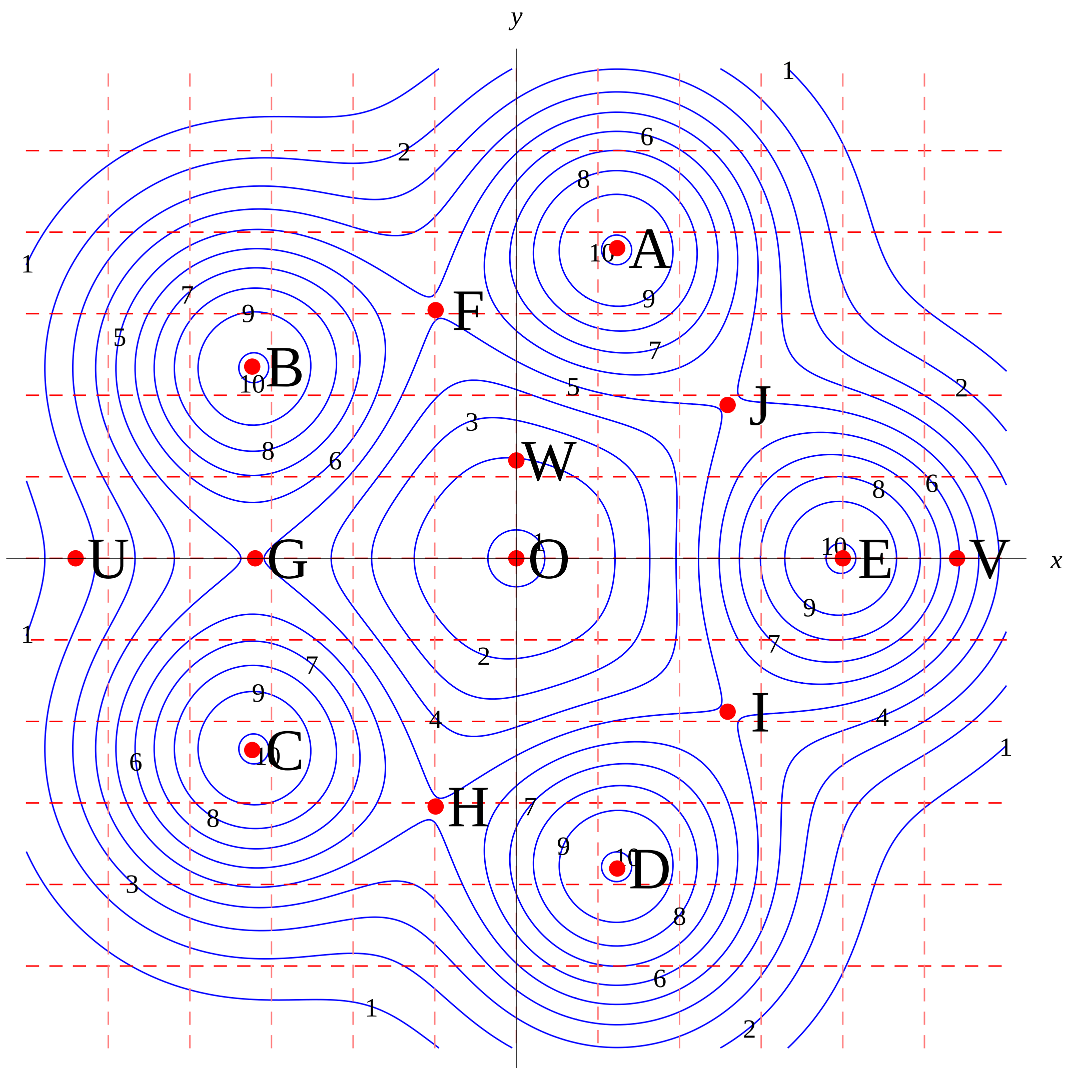
Problem 3) (10 points) (No justifications are needed.)

a) (6 points) A function $f(x, y)$ of two variables is shown as a contour map. Check what applies

	A	B	C	D	E	F	G	H	I	J	O	U	V	W
Local maximum														
Local minimum														
Saddle point														
Maximal steepness among A-W														
$f_x = 0, f_y \neq 0.$														
$f_y = 0, f_x \neq 0.$														

b) (4 points) Answer the following 4 questions.

What is the maximal height f reached when walking straight from U to V?	
What is the minimal height f reached when walking straight from U to V?	
There is a path from G to F on which height f is constant. True or False?	
There is a path from A to B on which height f is constant. True or False?	



Solution:

	A	B	C	D	E	F	G	H	I	J	O	U	V	W
Local maximum	*	*	*	*	*									
Local minimum											*			
Saddle point						*	*	*	*	*				
Maximal steepness among A-W													*	
$f_x = 0, f_y \neq 0$.														*
$f_y = 0, f_x \neq 0$.												*	*	

What is the maximal height f reached when walking straight from U to V?	10 at E
What is the minimal height f reached when walking straight from U to V?	0 at O
There is a path from G to F on which height f is constant. True or False?	True
There is a path from A to B on which height f is constant. True or False?	False

Problem 4) (10 points)

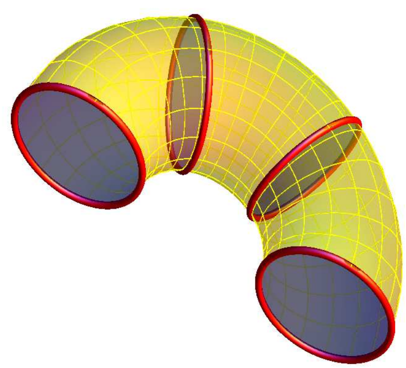
Maximize the volume of the half torus

$$f(x, y) = \pi^2 x^2 y$$

if the surface area of the torus including two disc dividers

$$g(x, y) = 4\pi x^2 + 2\pi^2 xy = \pi$$

is fixed.



Solution:
The Lagrange equations are

$$\begin{aligned} 2xy\pi^2 &= \lambda(8\pi x + 2\pi^2 y) \\ \pi^2 x^2 &= \lambda(2\pi^2 x) \\ 4\pi x^2 + 2\pi^2 xy &= \pi \ . \end{aligned}$$

Getting rid of $\lambda$from the first two equations gives $y = 4x/\pi$. Plugging this into the third equation allows to solve for x, y:

$$x = \frac{1}{\sqrt{12}}, y = \frac{2}{\sqrt{3}\pi} \ .$$

The maximal volume is $\boxed{\pi/(6\sqrt{3})}$.

Problem 5) (10 points)

Find all the critical points of the function

$$f(x, y) = x^3 - 3x + 1 + y^3 - 12y$$

and classify them. Also note whether any global maxima or minima are present.

Solution:

x	y	D	f_{xx}	Type	f
-1	-2	72	-6	maximum	19
-1	2	-72	-6	saddle	-13
1	-2	-72	6	saddle	15
1	2	72	6	minimum	-17

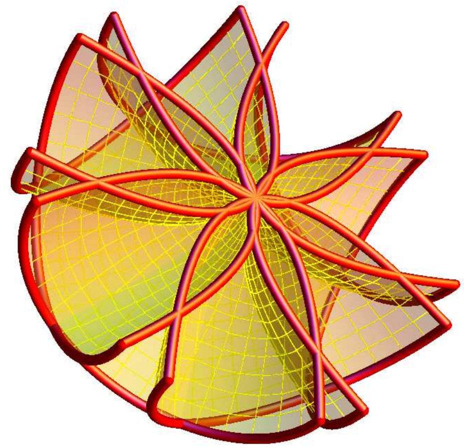
There are neither global maxima, nor global minima as for $y = 0$ already the function becomes unbounded above and below.

Problem 6) (10 points)

For an art project, we build a flower with 7 petals in which each petal is a paraboloid parametrized by

$$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), u^2/2\rangle$$

with $0 \leq v \leq \pi$ and $0 \leq u \leq 1$. Find the surface area of the flower. You have to find the area A of one petal and multiply by 7.



Solution:
In order to find the surface area $\int\int_R |r_u \times r_v| \, dudv$, we first compute

$$\vec{r}_u \times \vec{r}_v = \langle \cos(v), \sin(v), u\rangle \times \langle -u\sin(v), u\cos(v), 0\rangle = \langle -u^2\cos(v), -u^2\sin(v), u\rangle$$

which has length $\sqrt{u^2+u^4} = u\sqrt{1+u^2}$. We integrate

$$7\int_0^\pi \int_0^1 u\sqrt{u^2+1} \, dudv = \frac{7\pi}{3}(u^2+1)^{3/2}|_0^1 = \frac{7\pi}{3}(2^{3/2}-1) \ .$$

The final answer is $\boxed{\frac{7\pi}{3}(2^{3/2}-1)}$.

Problem 7) (10 points)

a) (5 points) Estimate $f(0.003, 0.98) = \sin(0.003) \cdot 0.98^5 + 3 \cdot 0.98$ by linear approximation.

b) (3 points) Find $D_{\vec{v}}f(0,1)$ if $\vec{v} = \langle 1,1\rangle/\sqrt{2}$.

c) (2 points) If $\vec{r}(t) = \langle t, 1-t\rangle$ is a path, express

$$\frac{d}{dt} f(\vec{r}(t))$$

at $t=0$ as a dot product of a vector with the vector $\vec{r}\,'(0) = \langle 1,-1\rangle$.

Solution:
a) The function

$$f(x,y) = \sin(x)y^5 + 3y$$

has the gradient

$$\nabla f(x,y) = \langle \cos(x)y^5, 5\sin(x)y^4 + 3\rangle$$

which is at $(0,1)$ equal to

$$\nabla f(0,1) = \langle 1, 3\rangle \ .$$

We have therefore

$$L(x,y) = 3 + 1(0.003) + 3(-0.02) = 2.943 \ .$$

The actual value is 2.94271. The linearization $\boxed{2.943}$ got the first three digits right.
b) This is a definition. The directional derivative is the scalar

$$D_{\vec{v}}f(0,1) = \langle 1, 3\rangle\langle 1, 1\rangle/\sqrt{2} = 4/\sqrt{2} \ .$$

The answer is $\boxed{2\sqrt{2}}$.
c) By the chain rule, we have

$$\vec{r}\,'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = \langle 1, 3\rangle \cdot \langle 1, 1\rangle = -2 \ .$$

The answer is $\boxed{\langle 1, 3\rangle \cdot \langle 1, 1\rangle = -2}$.

Problem 8) (10 points)

a) (5 points) Find the tangent line to $x + 2y^2 = 3$ at the point $P = (1,1)$.

b) (5 points) Find the tangent plane to $x^2 - y^2 + z^2 = 4$ at the point $(1,1,2)$.

Solution:
a) $\nabla f(x,y) = \langle 1, 4y\rangle$ and $\nabla f(1,1) = \langle 1, 4\rangle$. The tangent line is $x + 4y = d$, where d can be obtained by plugging in the point $(1,1)$. The solution is
$\boxed{x + 4y = 5}$. b) $\nabla f(x,y,z) = \langle 2x, -2y, 2z\rangle$ and $\nabla f(1,1,2) = \langle 2, -2, 4\rangle$. The tangent plane is $2x - 2y + 4z = d$ where d can be obtained by plugging in the point $(1,1,2)$. The solution is
$\boxed{2x - 2y + 4z = 8}$.

Problem 9) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y} \, dydx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2 + y^2) \, dydx \; ,$$

where R is the ring $1 \leq x^2 + y^2 \leq 9$.

Solution:

a) To change the order of integration, make a figure. The integration region is the upper left triangle in the square $[0, \pi/2] \times [0, \pi/2]$. We get

$$\int_0^{\pi/2} \int_0^y \cos(y)/y \, dxdy = \int_0^{\pi/2} \cos(y) \, dy = \sin(y)_0^{\pi/2} = 1 \; .$$

The answer is $\boxed{1}$.

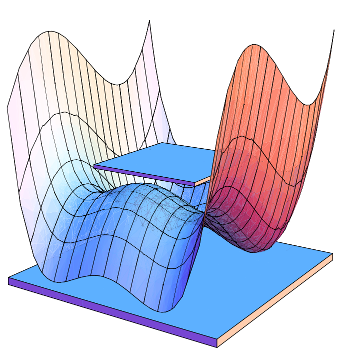
b) This is a polar integration problem

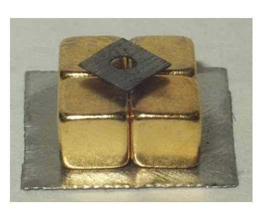
$$\int_0 0^{2\pi} \int_1^3 r^2 r \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4}|_1^3 \, d\theta = \int_0^{2\pi} 20 \, d\theta = 40\pi \; .$$

The answer is $\boxed{40\pi}$.

Problem 10) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form $f(x, y) = x^4 + y^3 - 2x^2 - 3y$. Find all critical points of this function and classify them. Is there a global minimum?





Right picture credit: Wikipedia.

Solution:
To find the critical points, we have to solve the system of equations $f_x = 4x^3 - 4x = 0, f_y = 3y^2 - 3 = 0$. The first equation gives $x = 0$ or $x = \pm 1$. The second equation $f_y = 3y^2 - 3 = 0$ gives $y = \pm 1$. There are $3 \cdot 2 = 6$ critical points. We compute the discriminant $D = 6y(12x^2 - 4)$ and $f_{xx} = 12x^2 - 4$ at each of the 6 points and use the second derivative test to determine the nature of the critical point.

point	D	$f_x x$	nature	value
(-1, -1)	-48	8	saddle	1
(-1, 1)	48	8	min	-3
(0, -1)	24	-4	max	2
(0, 1)	-24	-4	saddle	-2
(1, -1)	-48	8	saddle	1
(1, 1)	48	8	min	-3

There is no global minimum, nor any global maximum since for $x = 0$, the function is $f(0, y) = y^3 - 3y$ which is unbounded from above and from below (it goes to $\pm\infty$ for $y \to \pm\infty$).