7/24/2014 SECOND HOURLY PRACTICE I Maths 21a, O.Knill, Summer 2014

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F — The partial differential equation $u_y(x, y) = u_{xx}(x, y)$ is called the heat equation.

Solution:
Knowledge question. Even so, we use different variables

2) **T** F — If $(0, 0)$ is not a critical point of $f(x, y)$, then the level curve through $(0, 0)$ intersects a sufficiently small circle $x^2 + y^2 = r^2$ in exactly two points.

Solution:
Yes, we can see this intuitively or by linearization. When looking close at a point, the level curve is a line.

3) T **F** — A point (x_0, y_0) at which the gradient is zero, is called a discriminant.

Solution:
This is nonsense. Discriminant deals with the second derivative

4) T **F** — The relation $f_{xx} = f_{yy}$ holds everywhere as a consequence of Clairot's theorem.

Solution:
Clairot's theorem deals with mixed derivatives f_{xy}.

5) **T** F — The tangent plane to any point of the surface $(x + y + z)^2 = 4$ is either $x + y + z = 2$ or $x + y + z = -2$.

Solution:
The surface consists of two planes. The tangent plane is one of the planes.

6) T **F** — Fubini's theorem implies that for any function $f(x, y)$ of two variables, we have $\int_1^2 \int_3^4 f(x, y)\, dxdy = \int_3^4 \int_1^2 f(x, y)\, dxdy$.

Solution:
We have to switch also $dxdy$.

7) T **F** Saddle points with positive discriminant are called monkey saddles.

Solution:
No. This is monkey business.

8) **T** F If $f(x,y) = 1$ and R is a region in the plane, then $\int\int_R f(x,y)\; dxdy$ is the area of the region.

Solution:
Yes. This can be seen as the definition of area.

9) **T** F There is a function $f(x,y)$ for which the linearization $L(x,y)$ at a point is the function itself.

Solution:
Yes a linear function

10) T **F** The directional derivative at a local maximum is negative in every direction.

Solution:
It is zero

11) **T** F If $\vec{r}(t)$ is a curve on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Use the chain rule and the fact that $g(\vec{r}_(t))$ is constant so that $d/dt g(\vec{r}_(t))$ is zero.

12) T F If not at a critical point, the length $|\nabla f(0,0)|$ of the gradient is equal to the directional derivative into the direction of the gradient.

Solution:
Yes, we have seen this computation

13) T F If $D > 0$ and $\nabla f(0,0) = 0$ and $f_{xx} < 0$ then $f_{yy} < 0$.

Solution:
We have seen this from $D = f_{xx} f_{yy} - f_{xy}^2$.

14) T F $\int_0^1 \int_0^x f(x,y)\, dydx = \int_0^1 \int_y^1 f(x,y)\, dydx$.

Solution:
This is almost a correct correct switch but the $dxdy$ have also to be switched.

15) T F The surface area of the sphere of radius L was written in class as $\int_0^{2\pi} L^2 \sin(\phi)\, d\phi$.

Solution:
The answer would be 4π but the current expression is 0.

16) T F If $f(x,y)$ depends on one variable only, then the discriminant D satisfies $D = 0$ at every critical point.

Solution:
Indeed, then $f_{xy}, f_{xx} f_{yy}$ are both zero.

17) T F The gradient vector $\nabla f(x_0, y_0)$ is a vector in the plane perpendicular to the level curve of $f(x,y)$ through (x_0, y_0).

Solution:
Yes, here we go.

18) T F We have seen that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Solution:
We have seen this in class.

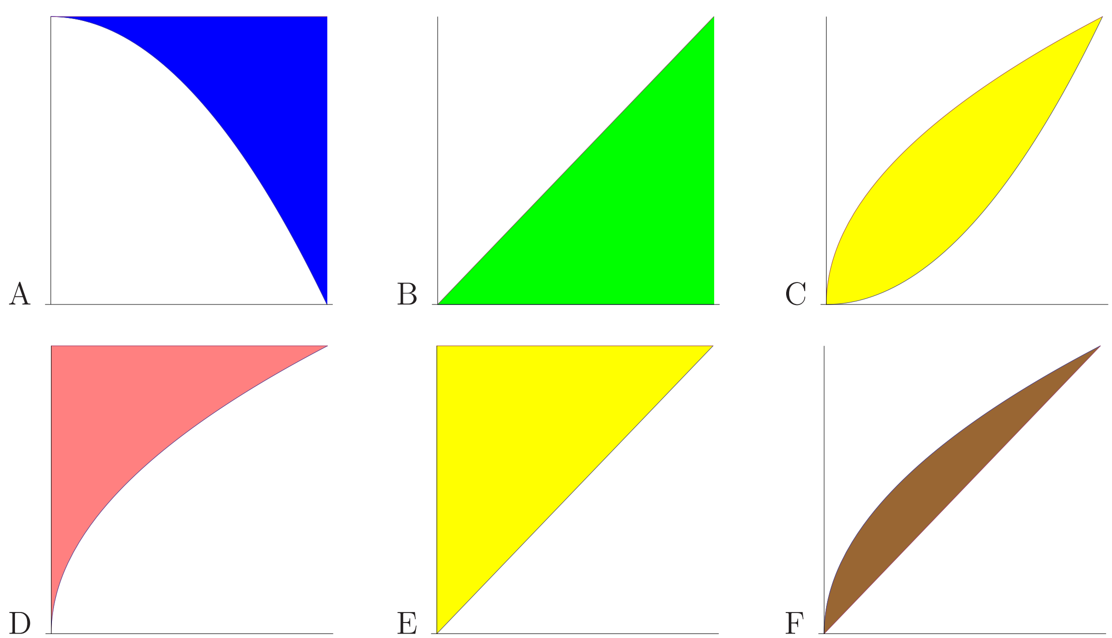
19) T F In three dimensions, the gradient of f has always the form $\nabla f(x, y) = \langle f_x, f_y, 1 \rangle$

Solution:
The gradient of f is a vector with two components.

20) T F If $f_{xx}(x, y) = f_{yy}(x, y)$ everywhere, then every critical point is a saddle point.

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A - F$.



Enter A-F	Integral
	$\int_0^1 \int_y^1 f(x,y)\ dxdy$
	$\int_0^1 \int_x^1 f(x,y)\ dydx$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\ dydx$
	$\int_0^1 \int_{y^2}^{y} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{y^2} f(x,y)\ dxdy$
	$\int_0^1 \int_{1-x^2}^{1} f(x,y)\ dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

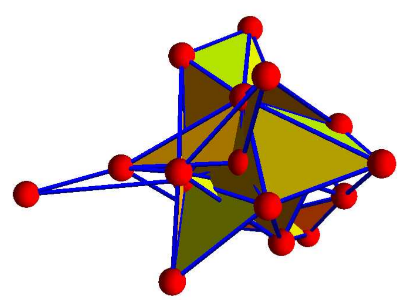
Fill in 1-4	Name
	Wave
	Transport
	Burgers
	Heat

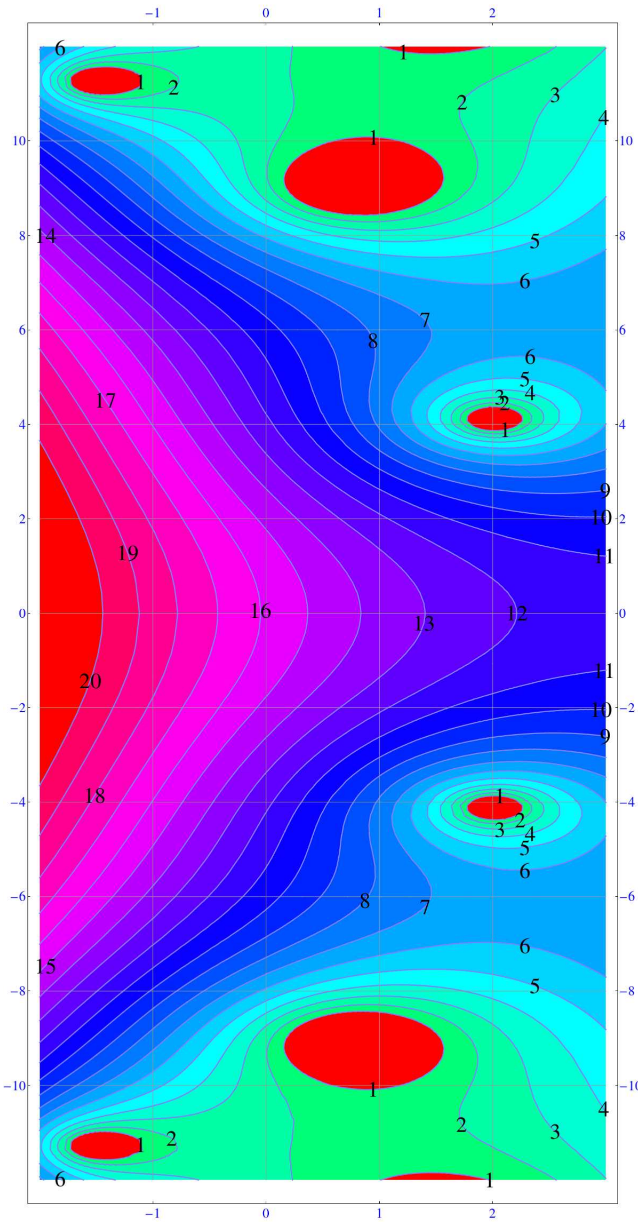
Equation Number	PDE
1	$u_x - u_y = 0$
2	$u_{xx} - u_{yy} = 0$
3	$u_x - u_{yy} = 0$
4	$u_y + uu_x - u_{xx} = 0$

Solution:
a) BECFDA
b) 2143

Problem 3) (10 points) (No justifications are needed.)

This summer, Oliver studies zeta functions of networks, functions of two variables which contain information about the network. They encode for example the frequencies at which the network resonates, if it is excited. Zeta functions are defined for geometric objects like surfaces. For a circle, it is the Riemann zeta function $f(x,y)$ for which finding the roots is a prize of 1 million dollars written out. The contour map below belongs to the zeta function of the network seen to the right. Check each box which is true. Don't split hairs. If something is not true, it should be clearly false.





- [ ] The point $(1, 10)$ is a local maximum.
- [ ] The point $(2, 4)$ is a local minimum.
- [ ] Close to $(2, -6)$, there is a saddle point.
- [ ] The partial derivative $f_x(0, 0)$ is positive.
- [ ] Under the constraint $x = 1$, f has a maximum near $y = 0$.
- [ ] Under the constraint $y = 8$, f has has a minimum near $x = 1$.
- [ ] The gradient at $(0, -8)$ is longer than the one at $(2, 8)$.
- [ ] The global maximum of f is at the boundary.
- [ ] Near $(1, -10)$, we have $D_{\langle 1,0 \rangle} f = 0$.
- [ ] Near $(1, -3)$, we have $D_{(1,1)/\sqrt{2}} f = 0$.

Solution:
Everything is right except the first and fourth entry.

Problem 4) (10 points)

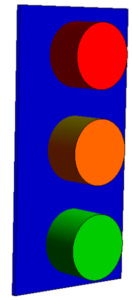
The material to build a traffic light is

$$g(x,y) = 6 + 6\pi xy + 3\pi x^2 = 12$$

is fixed (the radius of each cylinder is x and the height is y and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume

$$f(x,y) = 3\pi x^2 y$$

is maximal. Use the Lagrange method.



Solution:
The Lagrange equations $\nabla f = \lambda \nabla g, g = 12$ are

$$\begin{aligned} 6\pi xy &= \lambda(6\pi y + 6\pi x) \\ 3\pi x^2 &= \lambda(6\pi x) \\ 2\pi xy + \pi x^2 &= 6 \end{aligned}$$

Eliminating λ from the first two equations gives $x = y$. Plugging into the constraint gives $\boxed{x = \sqrt{2/(3\pi)} = y}$

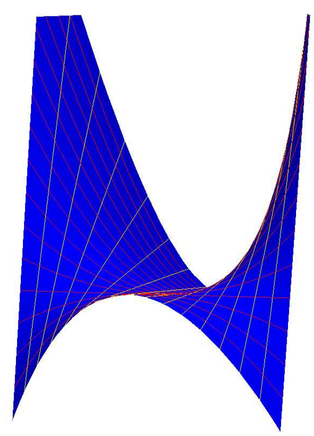
Problem 5) (10 points)

a) (8 points) Find all the critical points of the function

$$f(x,y) = xy - x - x^2y + x^2$$

and classify them.

b) (2 points) Are there global maxima and minima of $f(x,y)$?



Solution:
a) The gradient is $\nabla f(x,y) = \langle -1+2x+y-2xy, x-x^2 \rangle$. The second component being zero gives $x=0$ or $x=1$. In both cases, the first equation gives $y=1$. We also use $f_{xx} = 2-2y$, $f_{xy} = 1-2x$ so that $D = (1-2x)^2$ which is -1 in both cases. The function has two saddle points and no other critical point.

x	y	D	f_{xx}	Type	f
0	1	-1	0	saddle	0
1	1	-1	0	saddle	0

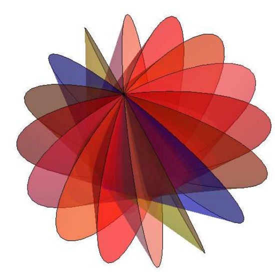
b) For $y=x$ the function is $2x^2 - x - x^3$ which has no upper, nor lower limit. There is no global maximum, nor global minimum. Besides establishing that the function is unbounded, one could also just say that if there would exist a global maximum or minimum at a finite point, then this would also have to be a local maximum rsp. minimum and would have shown up in part a).

Problem 6) (10 points)

A decorative paper lantern is made of 8 surfaces. Each is parametrized by

$$\vec{r}(t,z) = \langle 10z\cos(t), 10z\sin(t), z \rangle$$

with $0 \le t \le 2\pi$ and $0 \le z \le 1$ and then translated or rotated. Find the total surface area of the lantern.



Solution:
We compute

$$\begin{aligned}\vec{r}_t &= \langle -10z\sin(t), 10z\cos(t), 0\rangle \\ \vec{r}_z &= \langle 10\cos(t), 10\sin(t), 1\rangle \\ \vec{r}_t \times \vec{r}_z &= \langle 10z\cos(t), 10z\sin(t), -100z\rangle\end{aligned}$$

and the length is $|\vec{r}_t \times \vec{r}_z| = 10z\sqrt{101}$.

$$\int_0^{2\pi}\int_0^1 10z\sqrt{101}\ dzdt = 10\pi\sqrt{101}\ .$$

There are 8 pieces so that the final result is $\boxed{80\pi\sqrt{101}}$.

Problem 7) (10 points)

It is the year 2031, and you have been appointed royal "math magician" to **prince charming** just born to Kate, the Duchess of Cambridge and prince William. You show the teenage prince how to estimate

$$(\frac{126}{28})^{1/3}$$

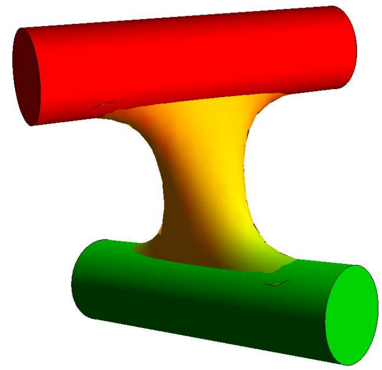
by finding a linearization of the function

$$f(x,y) = (x/y)^{1/3} = x^{1/3}y^{-1/3}$$

at the point $(x_0, y_0) = (125, 27)$. The young prince for a moment even ignores his iPhone 17 implanted in his skull and murmurs "thats soooh coool"!

Solution:
We have $f_x(x,y) = (1/3)x^{-2/3}y^{-1/3}$ which is $1/(3\cdot 25\cdot 3)$ $f_y(x,y) = (-1/3)x^{1/3}y^{-4/3} = -5/(3\cdot 3^4)$ $\boxed{5/3 + (1/225) - 5/243}$. This is $10027/6075 = 1.6505349794238683128....$ The actual number is 1.6509636244473133341. The estimate is 0.00042864502344 too low. Not bad. After the exam has gone to press, prince charming has got a name: **George Alexander Louis.** Now, 18 years old, the prince tells you that he had a hard time in school because they called him **GAL**.

Problem 8) (10 points)



a) (6 points) Find the tangent plane to the surface

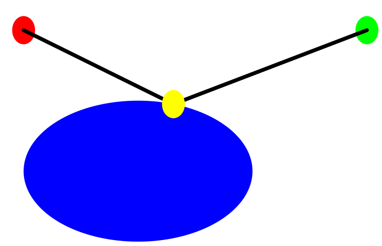
$$x^2 - xyz + y^2 + z^2 = 2$$

at the point $(1,1,1)$. The surface makes a nice connection piece between two cylinders.

b) (4 points) Find the tangent line to the curve $x^{1/3}y^{-1/3} = 1$ at $(1,1)$.

Solution:
a) $\nabla f(x,y,z) = \langle 2x - yz, 2y - xz, 2z - xy \rangle$ which is $\langle 1,1,1 \rangle$ at the point $(1,1,1)$. The equation is of the form $ax + by + cz = d$ where the constant d can be figured out by plugging in the point $\boxed{x + y + z = 3}$.
b) The gradient is $\nabla f(x,y) = \langle (1/3)x^{-2/3}y^{-1/3}, (-1/3)x^{-1/3}y^{-4/3} \rangle$ which is $\langle 1/3, -1/3 \rangle$ at the point $(1,1)$. The equation of the line is $x - y = d$ with some constant d. Plugging in the point gives $\boxed{x = y}$.

Problem 9) (10 points)



You find yourself in the desert at the point $A = (a,1)$, completely dehydrated and almost dead. You want to reach the point $B = (b,1)$ as fast as possible but you can not reach it without water. There is an lake inside the ellipsoid $g(x,y) = x^2 + 2y^2 = 1$. The amount of "effort" you need to go from a point (x,y) to a point (u,v) is assumed to be $(x-u)^2 + (y-v)^2$ (this is justified by the fact that if you walk for a long time, you walk less and less efficiently so that walking twice as long will take you 4 times as much effort). Find the path of least effort which connects A with $X = (x,y)$ and then with B.

a) Which function $f(x,y)$ do you extremize? The parameters a, b are constants.
b) Write down the Lagrange equations.

c) Solve the Lagrange equations in the case $a = -1, b = 1$.

Solution:
a) We have to extremize $f(x, y) = (x-a)^2 + (y-1)^2 + (x-b)^2 + (y-1)^2$ under the constraint $x^2 + 2y^2 = 1$.
b) The Lagrange equations are

$$\begin{aligned} 2(x-a) + 2(x-b) &= 2\lambda x \\ 4(y-1) &= 4\lambda y \\ x^2 + 2y^2 &= 1 \end{aligned}$$

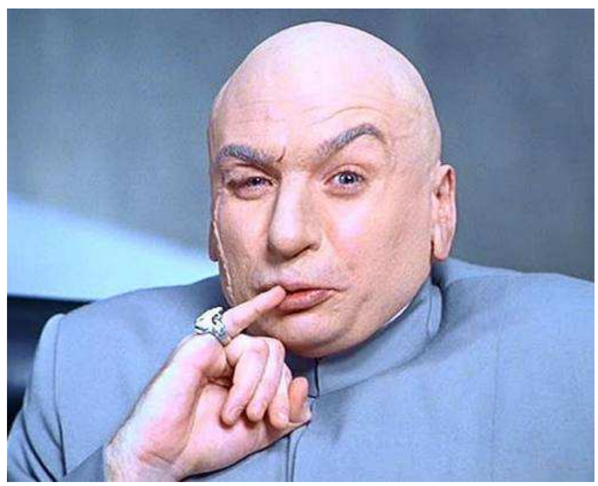
c) In the case $a = -1, b = 1$ we have extremal solutions $(0, 1/\sqrt{2})$ and $(0, -1/\sqrt{2})$. The first one is the minimum, the second the maximum.

Problem 10) (10 points)

a) (5 points) Integrate $f(x, y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \leq 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \; .$$



Solution:
a) Use polar coordinates:

$$\int_0^1 \int_0^{2\pi} (r^2\cos^2(\theta) - r^2\sin^2(\theta))r \; d\theta dr = \int_0^1 r^3 \; dr (\int_0^{2\pi} \cos(2\theta) \; d\theta) = (1/4)\cdot 0 = 0 \; .$$

The final answer is $\boxed{\text{zero}}$.
b) Write it in more convenient coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \; dydx \; .$$

This is a quarter disc in the x,y plane. Now use polar coordinates. The integral is evil because now, the θ, r have a different meaning. The integral in polar coordinates is

$$\int_0^1 \int_0^{\pi/2} r^2\sin^2(\theta)r \; d\theta dr$$

which is $\int_0^{\pi/2} \sin^2(\theta) \; d\theta \int_0^1 r^3 \; dr = (\pi/4)(1/4) = \pi/16$. (to compute the first integral, use the double angle formula $(1-\cos(2\theta))/2 = \sin^2(\theta)$.)