Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$.

Solution:
Yes, thats how we have formulated the chain rule.

2) | T | F | The point $(1, -1)$ is a critical point of $f(x, y) = x + y$.

Solution:
the function f has no critical point.

3) | T | F | The point $(0, 1)$ is a critical point of $f(x, y) = x$ under the constraint $g(x, y) = x^2 + y^2 = 1$.

Solution:
The points $(\pm 1, 0)$ are critical points but not $(0, 1)$: you can see this that the function value changes if we deviate on the circle from $(0, 1)$.

4) | T | F | The equation $u_y(x, y) = u_{yy}(x, y)$ is an example of an ordinary differential equation.

Solution:
Yes, there is only the derivative with respect to one variable, not two. In lecture, we have called this an ordinary differential equation.

5) | T | F | A point (x_0, y_0) at which the $D_{\langle 1,1 \rangle / 2^{1/2}}(x, y)$ is zero, is called a critical point.

Solution:
No, it could be zero without being a critical point. We have to have all directional derivatives to be zero.

6) | T | F | The relation $f_{xxyyxx} = f_{xyxyxy}$ holds everywhere for $f(x,y) = \sin(x^{10} + \cos(xy))$.

Solution:
Clairot's theorem deals with mixed derivatives but there are only three partial derivatives with respect to x,not 4.

7) | T | F | The tangent plane to a surface $z = x^2 + y^2$ at the point $(1,1,2)$ is given by $2x + 2y = 2$.

Solution:
No, we mixed up the solutions.

8) | T | F | Fubini's theorem and Clairot's theorem together imply $\int_0^1 \int_0^2 f_{xy}(x,y)\, dydx = \int_0^1 \int_0^2 f_{yx}(x,y)\, dxdy$.

Solution:
The integrand is the same by Clairot but we can not just change the order of the integration without changing the bound.

9) | T | F | A Monkey saddle point (x_0, y_0) of a function $f(x,y)$ has a negative discriminant D.

Solution:
No. it is zero there. You have seen the Monkey saddle in the homework. If you go around that critical point, there are three ups and downs, not two as in the usual saddle.

10) | T | F | $\int\int_R x^2 + y^2\, dxdy$ is the surface area of the paraboloid $z = x^2 + y^2$ located over the region R in the xy-plane.

Solution:
No, what the integral represents is a volume of the solid below the paraboloid surface and above the disc. It is not a surface area.

11) | T | F | If $f(0,0) = 0$ and the discriminant $D = 0$, then the linearization $L(x,y)$ of f at a point $(0,0)$ is constant zero.

Solution:
The discriminant D being zero does not imply that the gradient is zero.

12) | T | **F** | The directional derivative in the direction of the gradient is negative as it is the direction of steepest decent).

Solution:
It is positive as we have seen the formula $D_v f = |\nabla f|$ if $v = \nabla f / |\nabla f|$.

13) | **T** | F | If $\vec{r}(t)$ is a curve on the circle $g(x, y) = x^2 + y^2 = 1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Use the chain rule and the fact that $g(\vec{r}_(t))$ is constant so that $d/dt g(\vec{r}_(t))$ is zero.

14) | T | **F** | A point $(0, 0)$, where the discriminant D of f is maximal is a local minimum of f.

Solution:
Thats the biggest bull ever stated in a true false problem (since months). The discriminant being maximal does not have anything to do with the function having a minimum. To discredit the claim we have to give an example: take $f(x, y) = x^2 + y^2 - x^4 - y^4$. This function has a minimum at $(0, 0)$. The discriminant is $(2 - 12x^2)(2 - 12y^2)$ which has a local maximum at $(0, 0)$.

15) | T | **F** | If $f_{xx} > 0$, then the discriminant is always positive.

Solution:
No

16) | **T** | F | To maximize $f(x, y, z)$ under the constraint $g(x, y, z) = 1$, we have to solve the Lagrange equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, $g(x, y, z) = 1$.

Solution:
Yes, these are the Lagrange equations.

17) | T (highlighted) | F |

If $f(x,y) = \sin(x-y)$ then the discriminant D is zero at every critical point of f.

Solution:
Indeed, then $f_{xy}, f_{xx}f_{yy}$ are both zero.

18) | T | F (highlighted) |

The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the normal vector of the surface $z = f(x,y)$.

Solution:
Nonsense

19) | T | F (highlighted) |

If $|\nabla f(0,0)| = 10$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}} f(0,0) = -11$.

Solution:
The directional derivative is in absolute value always smaller than the length of the gradient.

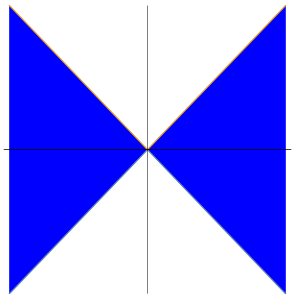
20) | T (highlighted) | F |

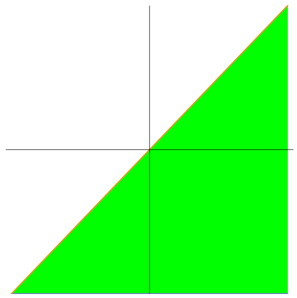
Assume $f(x,y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$.

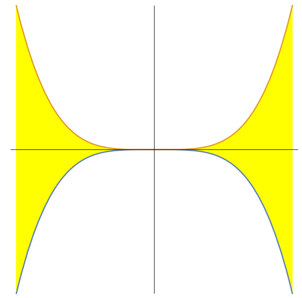
Solution:
This follows from the chain rule. It is called the gradient flow.

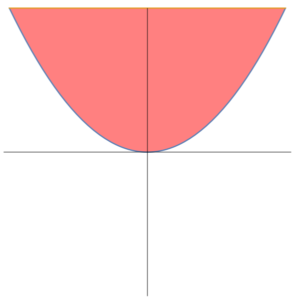
Problem 2) (10 points) No justifications are needed

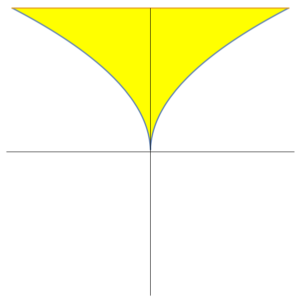
a) (6 points) Match the regions with the integrals. If no region matches, enter O.

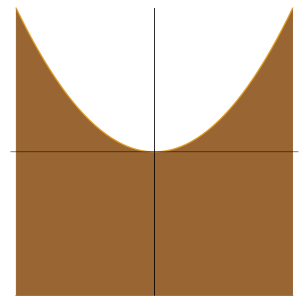
A


B


C


D


E


F


Enter A-F or O	Integral
	$\int_{-1}^{1}\int_{\lvert y\rvert}^{1} f(x,y)\ dxdy$
	$\int_{-1}^{1}\int_{x^3}^{x} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{-1}^{x} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{-x^4}^{x^4} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{\sqrt{\lvert x\rvert}}^{1} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{x^2}^{1} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{-1}^{x^2} f(x,y)\ dydx$
	$\int_{-1}^{1}\int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\ dydx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Wave
	Laplace
	Burgers
	Heat

Equation Number	PDE
1	$f_{\xi\xi} + f_{\eta\eta} = 0$
2	$f_{\eta} + f f_{\eta} - f_{\xi\xi} = 0$
3	$f_{\xi\xi} - f_{\eta\eta} = 0$
4	$f_{\xi} - f_{\eta\eta} = 0$

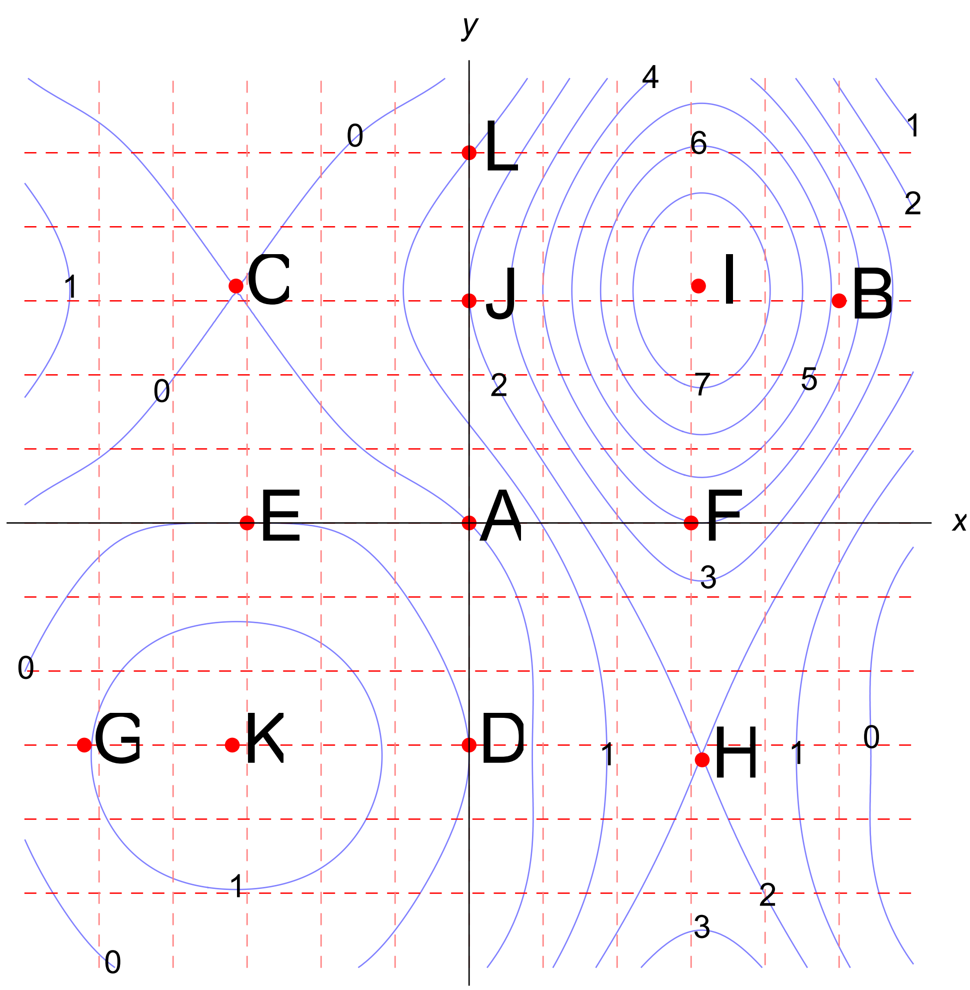
Solution:
a) OOBCEDFA
b) 3124

Problem 3) (10 points) (No justifications are needed.)

a) (5 points) You see a contour map of a function $f(x, y)$. Draw the gradient at each of the 5 points A-E. If the gradient should be zero, just mark the point with a bubble.

b) (5 points) Check the boxes which apply. It is in principle possible that more than one box has to be checked in a row or column or that no box needs to be checked in a row or column.

	A	B	C	D	E	F	G	H	I	J	K	L
Local maxima												
Local minima												
Saddle points												
Maximal steepness among A-L												
$f_x = 0, f_y \neq 0$												
$f_y = 0, f_x \neq 0$												
$D_{\langle 1,-1\rangle/2^{1/2}} f = 0$												
$D_{\langle 1,1\rangle/2^{1/2}} f = 0$												

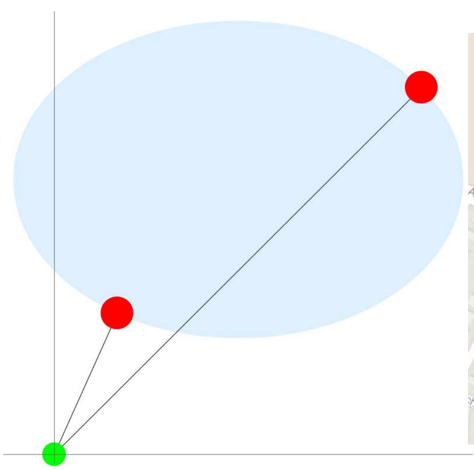


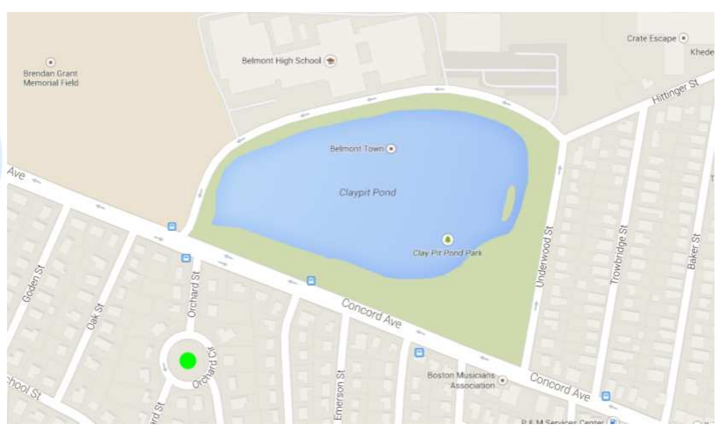
Solution:

	A	B	C	D	E	F	G	H	I	J	K	L
Local maxima									*		*	
Local minima												
Saddle points			*					*				
Maximal steepness among A-L		*										
$f_x = 0, f_y \neq 0$					*	*						
$f_y = 0, f_x \neq 0$		*		*			*			*		
$D_{\langle 1,-1\rangle/2^{1/2}} f = 0$	*		*					*	*		*	
$D_{\langle 1,1\rangle/2^{1/2}} f = 0$			*					*	*		*	*

Problem 4) (10 points)

Claypit pond near **Belmont high school** is a nice pond to run around. It has the shape $g(x,y) = (x-2)^2 + (y-3)^2 \leq 1$. Find the minimal and maximal distance of the "Orchard center" at $(0,0)$ to the pond. To do so, we find the maxima and minima of $f(x,y) = x^2 + y^2$ using Lagrange.





Solution:
The Lagrange equations are

$$\begin{aligned} 2x &= \lambda 2(x-2) \\ 2y &= \lambda 2(y-3) \\ (x-2)^2 + (y-3)^2 &= 1 \end{aligned}$$

The first two equations give $4x(y-3) = 4y(x-2)$ so that $3x = 2y$. Plugging $y = (3/2)x$ into the third equation gives a quadratic equation with $x = 2 \pm 2/\sqrt{13}$.

Problem 5) (10 points)

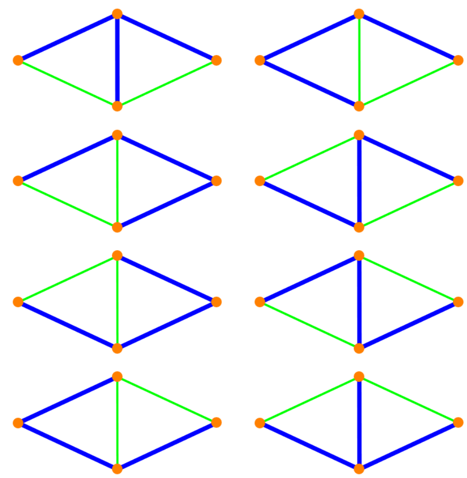
Graph theorists look at the **Tutte polynomial** $f(x,y)$ of a network. We work with the Tutte polynomial

$$f(x,y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**.

a) (4 points) Find the equations for the critical points an check that $(-2/3, 1/6), (0, -1/2)$ are solutions.

b) (6 points) Classify the two critical using the second derivative test.



Remark. The polynomial is useful: $xf(1-x,0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1,1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1,1) = 8$ as you see the 8 possible trees.

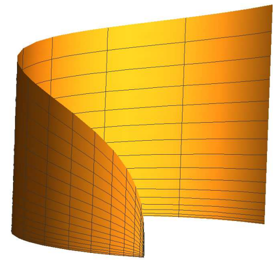
Solution:
The gradient is $\nabla f(x,y) = \langle 1 + 4x + 3x^2 + 2y, 1 + 2x + 2y \rangle$. The two given points are critical points. The point $(-2/3, 1/6)$ is a saddle point, the point $(0, -1/2)$ is a minimum. The discriminant at the first point is -4 at the second 4.

Problem 6) (10 points)

At the **Harvard graduate school of design**, a student constructs a wall parametrized by

$$\vec{r}(t,s) = \langle \sin(t^3), \cos(t^3), ts^2 \rangle$$

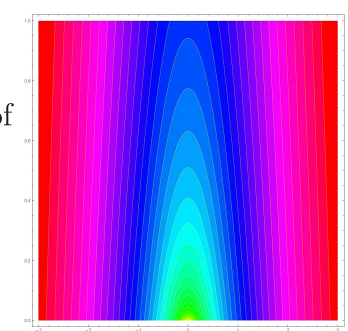
with $0 \leq t \leq 3$ and $0 \leq s \leq 1$. Find its surface area.



Remark: The upper figure shows the wall from the problem. Below you see an actual wall design from GSD photographed by Oliver in 2009 near GSD. By the way, the school is close to Memorial Hall. In the garden behind the school near the church, you can find still this interesting wall design.

Solution:
We have $|\vec{r}_t \times \vec{s}| = 6t^3 s$. $\int_0^3 \int_0^1 6t^3 s \; ds \; dt = 3^5/4 = 243/4$.

Problem 7) (10 points)



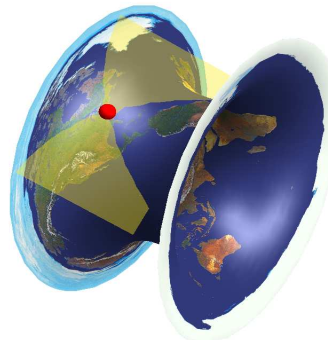
a) (6 points) Find the linearization $L(x,y)$ of $f(x,y) = (x^2+y)^{1/5}$ at $(x_0,y_0) = (32,0)$.

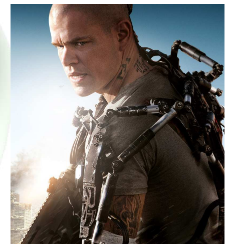
b) (4 points) Use this to estimate $(33^2+1)^{1/5}$.

Solution:
$L(x,y) = 4 + (1/20)(x-32) + (1/1280)y$. Evaluated at $(33,1)$ this is $L(33,1) = 4 + 1/20 + 1/1280$.

Problem 8) (10 points)

In the science fiction movie **Elysium** of 2013, humans have built a "paradise" for the rich in space. Inspired by this, we imagine an exact copy of the earth glued onto the surface $x^2 - y^4 + z^2 = 7$. Find the tangent plane at the point $(2, 1, 2)$ called "New-Boston". Our new hyperbolic world is for everybody. And imagine to see the North and South pole on the horizon! Great place to go climbing and skiing.

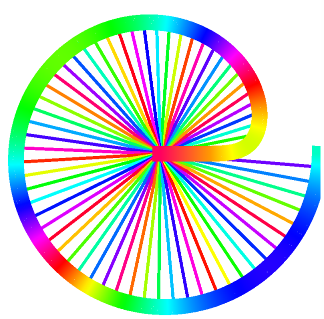


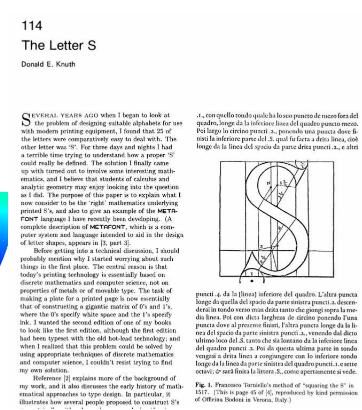


Solution:
The gradient is $\nabla f(x, y, z) = \langle 2x, -4y^3, 2z \rangle$. Evaluated at $(2, 1, 2)$ this is $\langle 4, -4, 4 \rangle$. The equation of the plane is $4x - 4y + 4z = d$. Plugging in the point gives $4x - 4y + 4z = 12$.

Problem 9) (10 points)

a) (5 points) We become typographer and design new mathematically defined **typeface** of the alphabet. The new letter "e" in this "21a" design is given by a polar region $r(t) \leq t^{1/7}$, with $0 \leq t \leq 2\pi$. Find the area of this region.





114

The Letter S

Donald E. Knuth

SEVERAL YEARS AGO when I began to look at the problem of designing suitable alphabets for use with modern printing equipment, I found that 25 of the letters were comparatively easy to deal with. The other letter was 'S'. For three days and nights I had a terrible time trying to understand how a proper 'S' could really be defined. The solution I finally came up with turned out to involve some interesting mathematics, and I believe that students of calculus and analytic geometry may enjoy looking into the question as I did. The purpose of this paper is to explain what I now consider to be the 'right' mathematics underlying printed S's, and also to give an example of the METAFONT language I have recently been developing. (A complete description of METAFONT, which is a computer system and language intended to aid in the design of letter shapes, appears in [3, part 3].

Before getting into a technical discussion, I should probably mention why I started worrying about such things in the first place. The central reason is that today's printing technology is essentially based on discrete mathematics and computer science, not on properties of metals or of movable type. The task of making a plate for a printed page is now essentially that of constructing a gigantic matrix of 0's and 1's, where the 0's specify white space and the 1's specify ink. I wanted the second edition of one of my books to look like the first edition, although the first edition had been typeset with the old hot-lead technology; and when I realized that this problem could be solved by using appropriate techniques of discrete mathematics and computer science, I couldn't resist trying to find my own solution.

Reference [2] explains more of the background of my work, and it also discusses the early history of mathematical approaches to type design. In particular, it illustrates how several people proposed to construct S's

.1., con quello tondo quale ha lo suo puncto de mezo fora del quadro, longe da la inferiore linea del quadro puncto mezo. Poi largo lo circino puncti .2., ponendo una puncta dove finisti la inferiore parte del .S. qual fu facta a drita linea, cioè longe da la linea del spacio da parte drita puncti .2., e altri

punci .4. da la [linea] inferiore del quadro. L'altra puncta longe da quella del spacio da parte sinistra puncti .2. descenderai in tondo verso man drita tanto che giongi sopra la media linea. Poi con dicta largheza de circino ponendo l'una puncta dove al presente finisti, l'altra puncta longe da la linea del spacio da parte sinistra puncti .2., venendo dal dicto ultimo loco del .S. tanto che sia lontano da la inferiore linea del quadro puncti .2. Poi da questa ultima parte in tondo vengasi a drita linea a congiungere con lo inferiore tondo longe da la linea da parte sinistra del quadro puncti .1. e sette octavi; & sarà finita la littera .S., come apertamente si vede.

Fig. 1. Francesco Torniello's method of "squaring the S" in 1517. (This is page 45 of [4], reproduced by kind permission of Officina Bodoni in Verona, Italy.)

b) (5 points) Integrate

$$\int_0^1 \int_0^{\arccos(y)} \frac{1}{\cos(x)} \, dxdy \ .$$

Remark: Computer scientist **Donald Knuth** once wrote an entire article about "The Letter S".

Solution:
a) We have $\int_0^{2\pi} t^{2/7}/2 \; dt = (7/18)(2\pi)^{9/7}$.
b) Change the order of integration. We have

$$\int_0^{\pi/2} \int_0^{y} \frac{1}{\cos(x)} \; dy dx = \int_0^{\pi/2} 1 \; dy = \pi/2 \; .$$