7/10/2014 FIRST HOURLY **Maths 21a, O.Knill, Summer 2014**

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Except for problems 1,2 and 6, give details.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T [✓ F] The integral $\int_a^b |T'(t)| \, dt$ measures the arc length of a curve.

Solution:
It is $r'(t)$ not $T'(t)$.

2) [✓ T] F The length of the vector projection $P_{\vec{v}}(\vec{w})$ is the absolute value of the scalar projection $\mathrm{comp}_{\vec{v}}(\vec{w})$.

Solution:
Yes according to the definition.

3) T [✓ F] The velocity vector of $\vec{r}(t) = \langle t^5, t^3, t^7 \rangle$ at time $t = 1$ is $\langle 1, 1, 1 \rangle$.

Solution:
It is $\langle 5, 3, 7 \rangle$.

4) T [✓ F] If $\vec{v} \times \vec{w} = \vec{v} \times \vec{w}$ then $\vec{v}, \vec{w}$ are parallel.

Solution:
The first statement is always true but the second not.

5) T [✓ F] The vector $\langle 1, 2, 3 \rangle$ is perpendicular to the line $\langle t, 2t, 3t \rangle$.

Solution:
No, it is parallel.

6) [✓ T] F The point given in spherical coordinates as $(\rho, \phi, \theta) = (1, \pi, \pi)$ is the same point than the point $(\rho, \phi, \theta) = (1, \pi, 0)$.

Solution:
Yes, it is the south pole.

7) [✓ T] F The parametrized curve $\vec{r}(t) = \langle 3\cos(t), 0, 5\sin(t)\rangle$ is an ellipse.

Solution:
Indeed, it is part of the xz-plane.

8) T [✓ F] The curvature of a line is 1 at every point of the line.

Solution:
No, it is 0.

9) T [✓ F] If $|\vec{v} \times \vec{w}| = \vec{v} \cdot \vec{w}$ then either $\vec{v}$ is parallel to $\vec{w}$ or perpendicular to $\vec{w}$.

Solution:
They can be unit vectors of angle 45 degrees for example

10) [✓ T] F If the dot product between two unit vectors is 1, then the two vectors are the same.

Solution:
Yes, $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\alpha) = \cos(\alpha) = 1$.

11) [✓ T] F We have $|(\vec{u} \times \vec{v}) \times \vec{w}| \leq |\vec{u}||\vec{v}||\vec{w}|$ for all vectors $\vec{v}, \vec{w}, \vec{u}$.

Solution:
Use the identity for the length of the cross product

12) T [✓ F] The curvature of a curve $\vec{r}(t)$ is given by $\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,''(t)|$.

Solution:
It is $\vec{r}'(t)$.

13) [✓ T] F The arc length of the curve $\langle \sin(t^2), \cos(t^2) \rangle$ from $t = 0$ to $t = 1$ is equal to 1.

Solution:
Yes, the speed is equal to 1 so that the integral is 1.

14) T [✓ F] Given three planes in space, there is a point which is on two planes at least.

Solution:
No, can all be parallel.

15) [✓ T] F The curve $\vec{r}(t) = \langle t^2, t, t^4 \rangle$ is the intersection curve of the two cylindrical paraboloids $x = y^2$ and $z = x^2$.

Solution:
Just plug in

16) [✓ T] F The lines $\vec{r}_1(t) = \langle 5 + t, 3 - t, 2 + t \rangle$ and $\vec{r}_2(t) = \langle 5 - t, 3 + t, 2t \rangle$ intersect perpendicularly.

Solution:
They have perpendicular velocity vectors but they do not appear to intersect. I had to regrade this as (thanks Changming for correcting me !) there is an intersection point $r_1(-2/3) = r_2(2/3)$. Wow this is a tricky one!

17) [✓ T] F You can intersect a two sheeted hyperboloid with a plane to get a parabola.

Solution:
Also this is trickier than anticipated. Its actually possible to get a parabola. Again thanks to Changming to point this out.

18) [✓ T] F The vector $\langle 12/13, 3/13, 4/13\rangle$ is a direction.

Solution:
Yes, its length is equal to 1.

19) T [✓ F] $\vec{v} \times (\vec{u} \times \vec{v}) = \vec{0}$ for all vectors $\vec{u}, \vec{v}$.

Solution:
Take $u = i, v = j$.

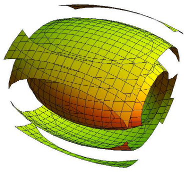
20) T [✓ F] If $f(x, y) = x^2y^2$, then $f_{xx} = f_{yy}$.

Solution:
Just differentiate. The equation $2y^2 = 2x^2$ is not true.

Total

Problem 2) (10 points) No justifications are needed in this problem.

a) (2 points) Match the contour plots. Enter O, if there is no match.



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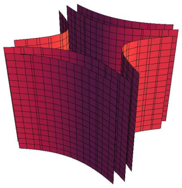
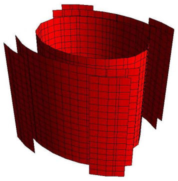


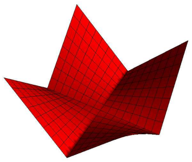
II



III

Function $g(x,y,z) =$	Enter O,I,II or III
$x + 2y$	
$x^2 + 2y^2 + 3z^2$	
$x^2 - 2y^2$	
$x^2 + 2y^2$	
$x^2 + 2y$	

b) (2 points) Match the graphs of the functions $f(x,y)$. Enter O, if there is no match.



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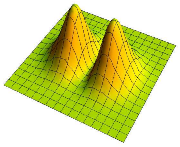

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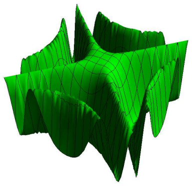
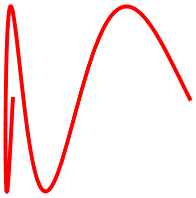


III

Function $f(x,y) =$	Enter O,I,II or III
$\cos(x^2 - y^2)$	
$\lvert xy \rvert$	
$x^2 \exp(-x^2 - y^2)$	
x^4	
$\exp(-x)$	

c) (2 points) Match the plane curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.



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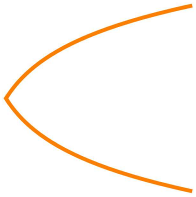

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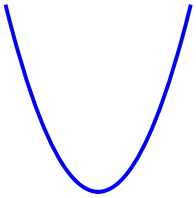
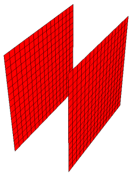


III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\vec{r}(t) = \langle t^2, 2t^2 \rangle$	
$\vec{r}(t) = \langle t^2, t^4 + 1 \rangle$	
$\vec{r}(t) = \langle \cos(\sqrt{t}), \sin(t) \rangle$	
$\vec{r}(t) = \langle t, t^3 \rangle$	
$\vec{r}(t) = \langle \exp(\lvert t \rvert), t \rangle$	

d) (2 points) Match functions g with level surface $g(x,y,z) = 1$. Enter O, if no match.



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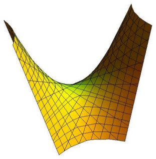

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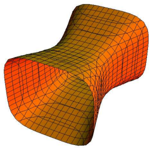
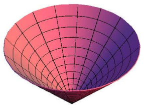


III

Function $g(x,y,z) = 1$	Enter O, I,II or III
$x^2 + y^4 = 1$	
$x^4 - y^4 + z^4 = 1$	
$y^4 = 1$	
$x^2 + y^2 + z^4 = 1$	
$xy - z = 1$	

e) (2 points) Match the parametrization. Enter O, where no match.



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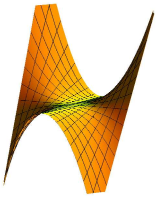

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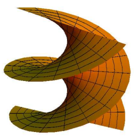


III

$\vec{r}(u,v) =$	Enter O,I,II or III
$\langle u, v, u^2 v \rangle$	
$\langle \cos v, uv, v \rangle$	
$\langle u\cos(v), u\sin(v), u \rangle$	
$\langle u\cos(v), u\sin(v), v \rangle$	
$\langle \cos v, u\sin(v), v \rangle$	

Solution:
a) O,I,II,III,O
b) III,I,II,O,O
c) O,III,I,O,II
d) O III,I,0,II
e) II,O,I,III,O

Problem 3) (10 points)

Routine ”toy” problems:

a) (2 points) $(\langle 1,1,1\rangle \cdot \langle 2,1,2\rangle) + |\langle 4,3,2\rangle|^2$.

b) (2 points) $|(\langle 2,3,4\rangle \times \langle 2,3,5\rangle) + \langle 0,6,0\rangle|$.

c) (2 points) $\langle 4,2,3\rangle \cdot (\langle 3,2,1\rangle \times \langle 1,1,1\rangle)$.

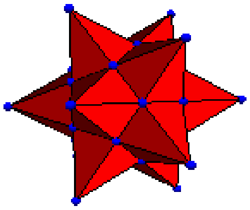
d) (2 points) $\vec{P}_{\vec{w}}(\vec{v})$ with $\vec{v} = \langle 1,4,1\rangle$ and $\vec{w} = \langle 9,18,9\rangle$.

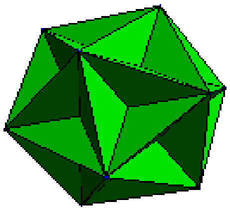
e) (2 points) The angle α between $\langle 0,3,3\rangle, \langle 2,0,2\rangle$.

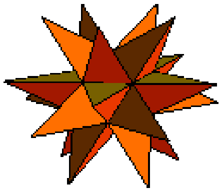
Solution:
a) 34
b) 5
c) 3
d) $\langle 5/3, 10/3, 5/3\rangle$.
e) $\pi/3$.

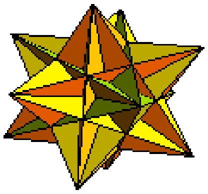
Problem 4) (10 points)

There are four **Kepler-Poinsot solids**. They are non-convex platonic solids. The first is centered at $A = (1,-1,1)$, the second at $B = (2,2,0)$, the third at $C = (4,5,6)$ the fourth at $P = (3,3,3)$. Find the distance from P to the plane through A, B, C.









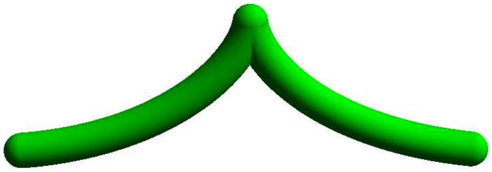
Solution:
This is a standard distance problem with $\vec{PA} = \langle 2, 4, 2\rangle$ and $\vec{n} = \langle 21, -8, -3\rangle$. The answer $\vec{PA} \cdot \vec{n}/|\vec{n}|$ is $4/\sqrt{514}$.

Problem 5) (10 points)

a) (7 points) A **cloth hanger** is padded around a curve which has the form

$$\vec{r}(t) = \langle t^2\cos(t), t^2\sin(t), t^3/3\rangle$$

with $-1 \leq t \leq 1$. Find the arc length from $t = -1$ to $t = 1$.
b) (3 points) As the picture indicates, the curvature appears problematic at $t = 0$. Verify, that the curvature does not exist there by writing down a formula for the curvature and indicating that it can not be evaluated at $t = 0$.



Solution:
Find $\int_{-1}^{1} |\vec{r}\ ;(t)|\,dt = \int_{-1}^{1}\sqrt{4t^2 + 2t^4}\,dt$. Take t^2 outside the square root and use substitution to solve the problem. Since the function $u = 4 + 2t^2$ is not one to one on $[-1, 1]$ one had to split up the integral and get $2\int_0^1 t\sqrt{4 + 2t^2}\,dt = 2\int_4^6 \sqrt{u}/4\,du = (1/3)u^{3/2}|_4^6 = 2\sqrt{6} - 8/3$.

Problem 6) (10 points)

No justifications are needed in this problem. You can assume that $\vec{r}(t)$ is a space curve, and that the vectors $\vec{v}, \vec{w}, \vec{r}\,', \vec{r}\,'', \vec{T}, \vec{T}\,'$ are non-zero, where $\vec{N}$ is the normal vector and $\vec{B}$ the bi-normal vector. All these vectors $\vec{r}, \vec{T}, \vec{B}, \vec{N}$ and its derivatives are evaluated at a fixed time, like $t = 0$.

First vector	Second vector	always parallel	always perpendicular	depends
$\vec{v} \times \vec{v}$	$\vec{v}$			
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$			
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$			
$\mathrm{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$			
$\mathrm{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			
$\vec{T}$	$\vec{r}\,'$			
$\vec{T}$	$\vec{T}\,'$			
$\vec{T}'$	$\vec{r}\,''$			
$\vec{B}$	$\vec{N}$			

Solution:

First vector	Second vector	always parallel	always perpendicular	depends
$\vec{v} \times \vec{v}$	$\vec{v}$	x	x	
$\vec{w} + \vec{v}$	$\vec{v} - \vec{w}$			x
$\vec{v} \times \vec{w}$	$\vec{w} \times \vec{v}$	x		
$(\vec{v} + \vec{w}) \times \vec{w}$	$\vec{v} \times \vec{w}$	x		
$\mathrm{Proj}_{\vec{v}}(\vec{w})$	$\vec{v}$	x		
$\mathrm{Proj}_{\vec{v}}(\vec{w})$	$\vec{w}$			x
$\vec{T}$	$\vec{r}\,'$	x		
$\vec{T}$	$\vec{T}\,'$		x	
$\vec{T}'$	$\vec{r}\,''$			x
$\vec{B}$	$\vec{N}$		x	

Problem 7) (10 points)

111 Huntington is a tall building in Boston which is in the shadow of the Prudential and Hancock tower. But its probably the most interesting building mathematically. It is only 169 meters high (compared with Hancock with 241) but look at the beauty of the dome! Inside this structure there are two metal connection lines

$$\vec{r}_1(t) = \langle t, 3t, -t \rangle$$

and

$$\vec{r}_2(t) = \langle 1 + 2t, t, t \rangle \ .$$

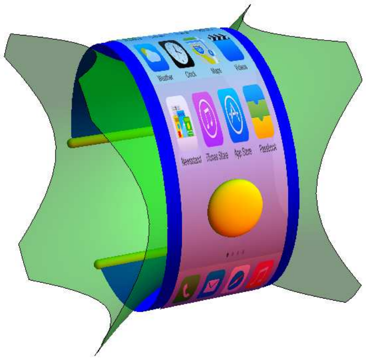
Find their distance.

Solution:
The point
The vector
$= \langle 4, -3,$

Problem 8) (10 points)

Very soon, Apple is expected to come out with an **iWatch**. Lets build one ourselves. Simplicity is the rule: it consists only of a band, and a "home button". And because Apple is known for fancy packaging, we strapped around a one sheeted hyperboloid. For each of the following surfaces, find a parametrization of the form

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \ .$$



a) (3 points) "Band" cylinder

$$x^2 + y^2 = 9 \ .$$

b) (4 points) "Button" ellipsoid

$$10(x - 3)^2 + y^2 + z^2 = 1/4 \ .$$

c) (3 points) "Package" 1-sheeted hyperboloid

$$x^2 + y^2 - z^2 - 1/2 = 0 \ .$$

Solution:

a) $\vec{r}(u,v) = \langle 3\cos(u), 3\sin(u), v\rangle$.
b) $\vec{r}(u,v) = \langle 3 + \frac{1}{\sqrt{40}}\cos(u)\sin(v), \frac{1}{2}\sin(u)\sin(v), \frac{1}{2}\cos(v)\rangle$.
c) $\vec{r}(u,v) = \langle \sqrt{v^2+1/2}\cos(u), \sqrt{v^2+1/2}\sin(u), v\rangle$.

Problem 9) (10 points)

This June 2014, the Swiss extreme sports women **Géraldine Fasnacht** jumped with a wing-suit from the Matterhorn (a mountain in Switzerland). When flying with a wingsuit, there is the gravitational force, the force from the wind and a force from the wing. Assume

$$\vec{r}''(t) = \langle 1, t, \exp(-t) - 10\rangle$$

and $\vec{r}(0) = \langle 0,0,4500\rangle$ and $\vec{r}\,'(0) = \langle 0,2,0\rangle$. Find the path $\vec{r}(t)$.

Solution:

Integrate twice and compare initial conditions: $\vec{r}(t) = \langle t^2/2, t^3/6+2t, e^{-t}-5t^2-t+4499\rangle$.