Lecture 9: Partial derivatives

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called the **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

We use the short hand notation $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. For iterated derivatives, the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. The meaning of $f_x(x_0,y_0)$ is the slope of the graph sliced at (x_0,y_0) in the x direction. The second derivative f_{xx} is a measure of concavity in that direction. The meaning of f_{xy} is the rate of change of the slope if you change the slicing.

The notation for partial derivatives $\partial_x f, \partial_y f$ was introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences". Partial derivatives f_x and f_y measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

1 For $f(x,y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**: it is an equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to more than one variables.

Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$$h f_x(x,y) = f(x+h,y) - f(x,y)$$
$$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x,y+h) - (f(x+h,y) - f(x,y))$$

$$h f_y(x,y) = f(x,y+h) - f(x,y).$$
$$h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$$

We have not taken any limits in this proof. We have established an identity which holds for all $h > 0$: the discrete derivatives f_x, f_y satisfy the relation $f_{xy} = f_{yx}$. We could fancy the identity obtained in the proof as a "quantum Clairaut" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, we can take the limit $h \to 0$ to get the classical Clairaut's theorem as a "classical limit". Note that the quantum Clairaut theorem shown first in this proof holds for **all** functions $f(x,y)$ of two variables. We do not even need continuity.

2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6 y^{10} \cos(y)$. Answer: Do not compute, but think.

3 The continuity assumption for f_{xy} is necessary. The example

$$f(x,y) = \frac{x^3y - xy^3}{x^2 + y^2}$$

contradicts Clairaut's theorem:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation** (PDE) If only the derivative with respect to one variable appears, it is an **ordinary differential equation** (ODE).

Here are examples of partial differential equations. You have to know the first four in the same way than a chemist has to know what $H_2O, CO_2, CH_4, NaCl$ is. Of course, as more to know, as better: rubber C_5H_8, Asprin $C_9H_8C_4$ Ethanol C_2H_6, Ammonia NH_3 etc.

4 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $f(t,x) = \sin(x-t) + \sin(x+t)$ satisfies the wave equation.

5 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

6 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x,y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation.

7 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t,x) = e^{-(x+t)^2}$ satisfies the advection equation.

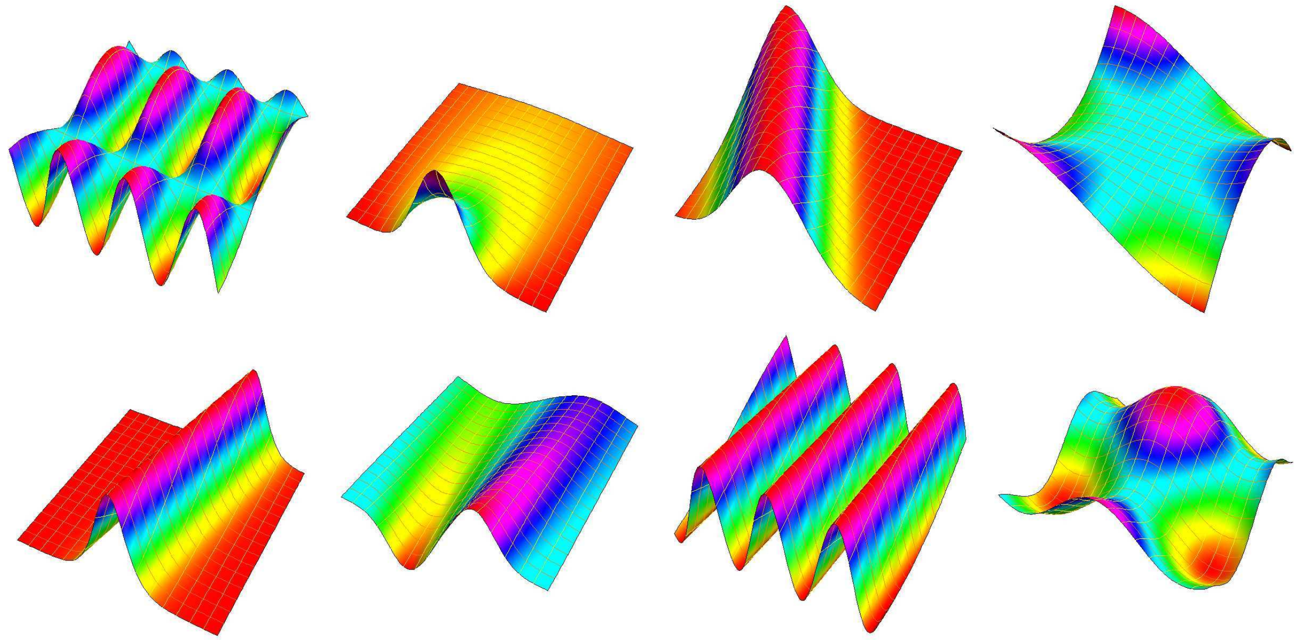
8 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $f(x,y) = \cos(x) + \sin(y)$ satisfies the eiconal equation.

9 The **Burgers equation** $f_t + ff_x = f_{xx}$ describes waves at the beach which break. The function $f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.

10 The **KdV equation** $f_t + 6ff_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2t))$ satisfies the KdV equation.

11 The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $f(t,x) = e^{i(kx - \frac{\hbar}{2m}k^2 t)}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Here are the graphs of the solutions of the equations. Can you match them with the PDE's?



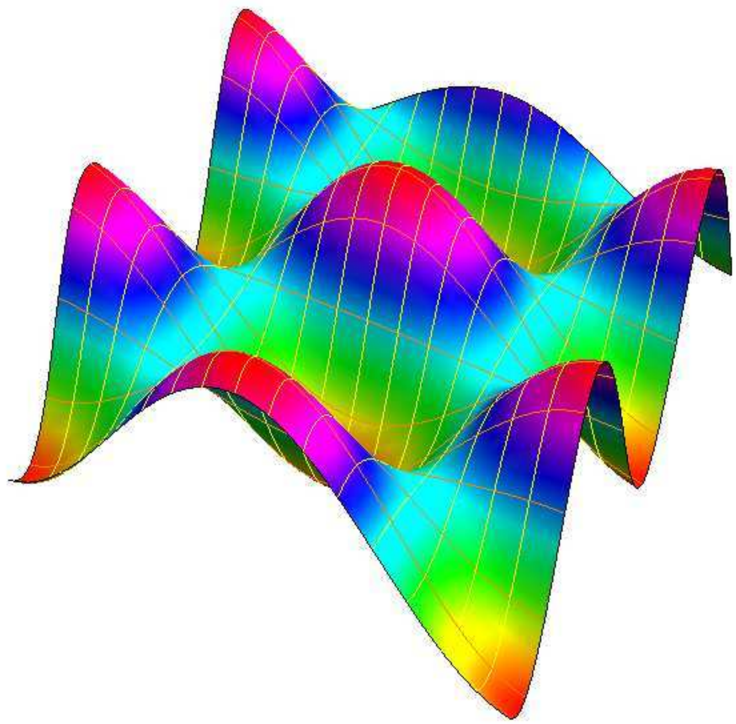
Notice that in all these examples, we have just given one possible solution to the partial differential equation. There are in general many solutions and only additional conditions like initial or boundary conditions determine the solution uniquely. If we know $f(0,x)$ for the Burgers equation, then the solution $f(t,x)$ is determined. A course on partial differential equations would show you how to get the solution.

Paul Dirac once said: "A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed. We will not learn here to find solutions to partial differential equations. But you should be able to verify that a given function is a solution of the equation.

Homework

1 Verify that $f(t,x) = \tan(\sin(t+x))$ is a solution of the transport equation $f_t(t,x) = f_x(t,x)$.

2 a) Verify that $f(x,y) = \cos(x)(\cos(2y) + \sin(2y))$ satisfies the **Klein Gordon equation** $u_{xx} - u_{yy} = 3u$. This PDE is useful in quantum mechanics.
b) Verify that more generally, $\cos(bx)(\cos(ay) + \sin(ay))$ satisfies the Klein gordon equation $u_{xx} - u_{yy} = (a^2 - b^2)u$.

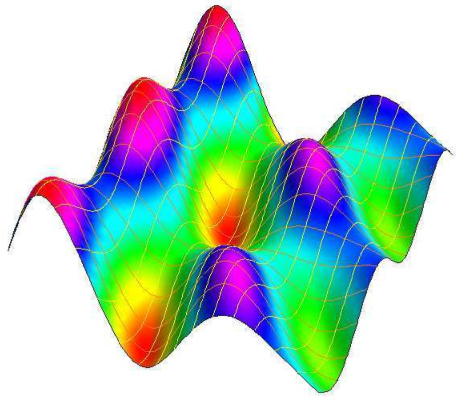


3 Verify that $f(x,t) = e^{-rt}\sin(x+ct)$ satisfies the driven transport equation $f_t(x,t) = cf_x(x,t) - rf(x,t)$ It is sometimes also called the **advection equation**.

4 The partial differential equation $f_{xx} + f_{yy} = f_{tt}$ is called the wave equation in two dimensions. It describes waves in a pool for example.

a) Show that if $f(x,y,t) = \sin(nx+my)\sin(\sqrt{n^2+m^2}\ t)$ satisfies the wave equation. It describes waves in a square where $x \in [0,\pi]$ and $y \in [0,\pi]$. The waves are zero at the boundary of the pool.

b) For which k is $f(x,y,t) = \sin(nx)\cos(nt)+\sin(mx)\cos(mt)+\sin(nx+my)\cos(kt)$ do we get solution of the wave equation which is periodic in time? You might want to know that integers m,n,k which satisfy $m^2+n^2 = k^2$ are called **Pythagorean triples**.



5 The partial differential equation $f_t + f f_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the Navier-Stokes equation which are used to describe the weather. Verify that the function

$$f(t,x)\frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}}e^{-\frac{x^2}{4t}}+1}$$

is a solution of the Burgers equation.

Remark. This calculation needs perseverance, when done by hand. You are welcome to use technology if you should get stuck. Here is an example on how to check that a function is a solution of the heat equation in Mathematica:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[D[f[t,x],t] == D[f[t,x],{x,2}]]
```

And here is the function

```
f[t, x] :=(1/t)^(3/2)*x*Exp[-(x^2)/(4t)]/((1/t)^(1/2)*E
```

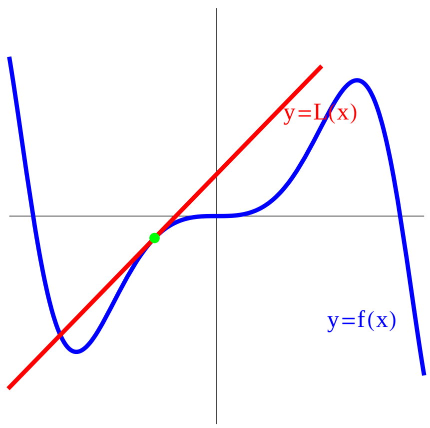
Lecture 10: Linearization

In single variable calculus, you have seen the following definition:

The **linear approximation** of $f(x)$ at a point a is the linear function

$$L(x) = f(a) + f'(a)(x - a) .$$

Its important to think about this in terms of functions and not graphs because for functions of three variables and more we can not draw graphs anymore.



The graph of the function L is close to the graph of f at a. We generalize this to higher dimensions:

The **linear approximation** of $f(x, y)$ at (a, b) is the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) .$$

The **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) .$$

Using the **gradient**

$$\nabla f(x, y) = \langle f_x, f_y \rangle, \qquad \nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .$$

How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x, b) = f(a, b) + f_x(a, b)(x - a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

1 What is the linear approximation of the function $f(x,y) = \sin(\pi x y^2)$ at the point $(1,1)$? We have $(f_x(x,y), f_y(x,y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = \langle \pi \cos(\pi), 2\pi \cos(\pi) \rangle = \langle -\pi, 2\pi \rangle$.

2 Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 .$$

3 Here is an example in three dimensions: find the linear approximation to $f(x,y,z) = xy + yz + zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = (2,2,2)$. we have $L(x,y,z) = f(1,1,1) + (2,2,2) \cdot (x-1, y-1, z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

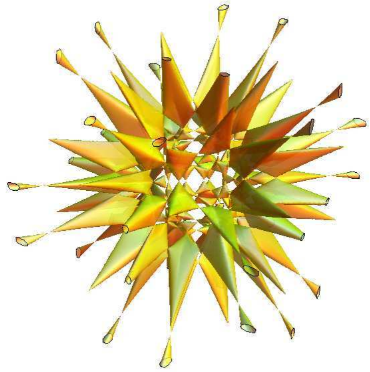
4 Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x \sqrt{y} z$.
Solution: take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

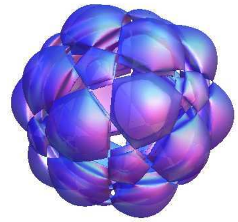
5 Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$.
Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $\langle a, b \rangle = \nabla f(2,4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

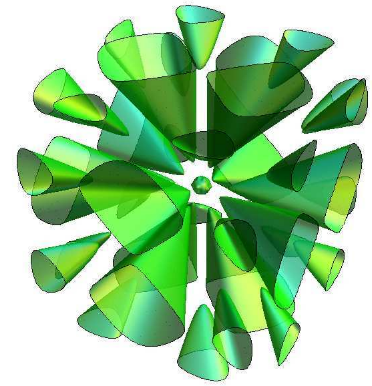
6 The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x,y,z) &= (3+5t)(-1+x^2+y^2+z^2)^2(-2+t+x^2+y^2+z^2)^2 \\ &+ 8(x^2 - t^4 y^2)(-(t^4 x^2) + z^2)(y^2 - t^4 z^2)(x^4 - 2x^2 y^2 + y^4 - 2x^2 z^2 - 2y^2 z^2 + z^4) , \end{aligned}$$

where $t = (\sqrt{5}+1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5}-1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x,y,z) = 8$. Find the tangent plane of the later surface at the point $(1,1,0)$. **Answer:** We have $\nabla f(1,1,0) = \langle 64, 64, 0 \rangle$. The surface is $x + y = d$ for some constant d. By plugging in $(1,1,0)$ we see that $x + y = 2$.



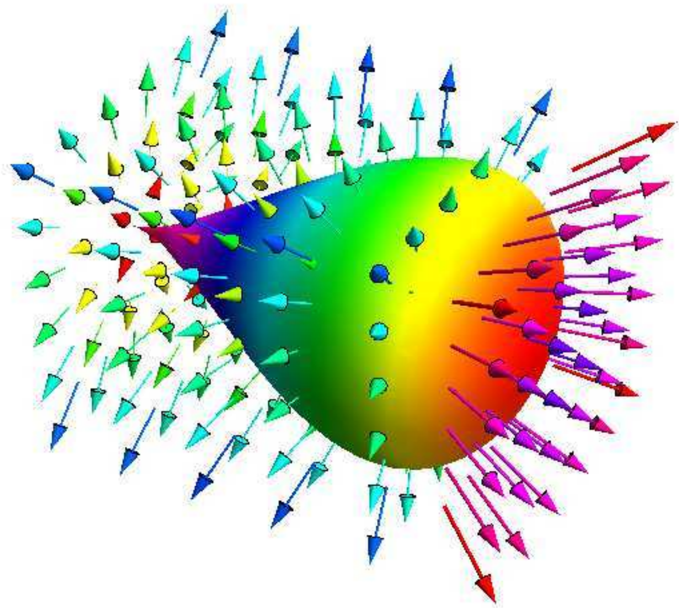




7 The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? We get $\langle a,b,c \rangle = \langle 20,4,4 \rangle$ and so the equation of the plane $20x + 4y + 4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



Remark: some traditional text books like Stewart use **differentials** to describe linearizations. We recommend to avoid this 19th century notation and terminology. Newton has used used terms like "fluxions", Leibniz "differentials", its time to move on. For us, the linearlization of a function f at a point is a **linear function** L in the same number of variables defined above. 20th century mathematics has invented the notion of **differential forms** which is an extremely valuable mathematical notion, but it is a concept which needs some multi-linear algebra and becomes only useful in more advanced topics like differential geometry or topology. Similarly, the notion of **infinitesimal small quantities** has been made precise in a language called nonstandard analysis but it needs a considerable amount of background knowledge in logic to be appreciated and understood. The notion of "differentials" comes from a time when calculus was still foggy in some areas. Unfortunately, the term has survived and appears even still in some calculus books. If you are not convinced by what was just said, try to find out (by looking on the web) what people mean with "differential": you find notions like "change in the linearization of a function" or "infinitesimals" which are both good examples of what is "foggy terminology". To add to the confusion, sometimes also terms like dx are called a "differentials". This appears in integrals $\int \sin(x)\, dx$ but note that in that case, it is just used as part of the **notation** to indicate with respect to which variable we integrate. Mathematica for example writes this as $Integrate[Sin[x], x]$ and there is no mystery. It just means to find the anti-derivative of $\sin(x)$ with respect to x.

Homework

1 Given $f(x,y) = \sin(x) - yx/\pi$. Estimate $f(\pi + 0.01, 0.97)$ using linearization

2 Estimate $10'000'000^{1/10}$ using linear approximation.

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3 Estimate $f(0.01, 0.9999)$ for $f(x,y) = \cos(\pi xy)y + \sin(x + \pi y)$ using linearlization.

4 Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

5 Sketch a contour map of the function

$$f(x,y) = x^2 + 9y^2$$

find the **gradient vector** $\nabla f = \langle f_x, f_y \rangle$ of f at the point $(1,1)$. Draw it together with the tangent line $ax + by = d$ to the curve at $(1,1)$.

Lecture 11: Chain rule

If f and g are functions of a single variable t, the **single variable chain rule** tells us that $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.
It can be proven by linearizing the functions f and g and verifying the chain rule in the linear case. The **chain rule** is also useful:

For example, to find $\arccos'(x)$, we differentiate $x = \cos(\arccos(x))$ to get $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x))\arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))}\arccos'(x) = -\sqrt{1-x^2}\arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

If $\vec{r}(t)$ is curve and f is a function of several variables we can build a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, If $\vec{r}(t)$ is a parametrization of a curve in the plane and f is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

The **multivariable chain rule** is $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$.

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t),y(t)) = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t) .$$

Now, the identity

$$\frac{f(x(t+h),y(t+h))-f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h))-f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h))-f(x(t),y(t))}{h}$$

holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t),y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate f with respect to t and y is treated as a constant:

$$\frac{f(\text{x(t+h)}) - f(x(t))}{h} = \frac{[f(\text{x(t)} + \text{(x(t+h)-x(t))}) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} .$$

Write $H(t) = $ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t)+H) - f(x(t))]}{H} \cdot \frac{x(t+h)-x(t)}{h} .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.

1 We move on a circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ on a table with temperature distribution $f(x, y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x, y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

From $f(x, y) = 0$ one can express y as a function of x. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain $y' = -f_x/f_y$. Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x, y, z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x, y)$. We can compute the derivative z_x without actually knowing the function $z(x, y)$. To do so, we consider y a fixed parameter and compute using the chain rule

$$f_x(x, y, z(x, y))1 + f_z(x, y)z_x(x, y) = 0$$

so that $z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$.

2 The surface $f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x, y)$ at the point $(x, y, z) = (2, 1, 1)$.
Solution: $z_x(x, y) = -f_x(2, 1, 1)/f_z(2, 1, 1) = -4/(2/9) = -18$.

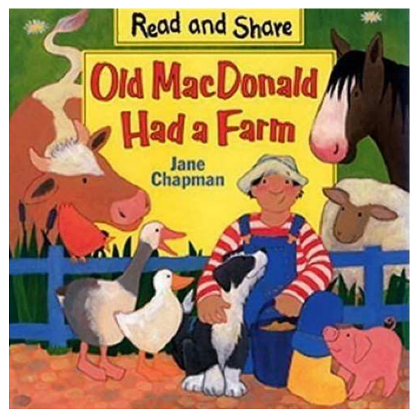
The chain rule is powerful because it implies other differentiation rules like the addition, product and quotient rule in one dimensions: $f(x, y) = x + y, x = u(t), y = v(t), d/dt(x + y) = f_x u' + f_y v' = u' + v'$.
$f(x, y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x, y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.
As in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x, y) = ax + by - c$ and if the curve $\vec{r}(t) = \langle x_0 + tu, y_0 + tv \rangle$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a, b)$ with $\vec{r}\,'(t) = (u, v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

Homework

1 You know that $d/dt f(\vec{r}(t)) = 3$ at $t = 0$ if $\vec{r}(t) = \langle t, t \rangle$ and $d/dt f(\vec{r}(t)) = 5$ at $t = 0$. $\vec{r}(t) = \langle t, -t \rangle$. Find the gradient of f at $(0, 0)$.

2 The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

3 The chain rule is closely related to linearization as it could be proven by linearlization. Lets get back to linearlization a bit: A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y. Estimate $f(12, 19)$.



P.S. In the fall of 2013, Oliver made a song out of this:

"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"

4 Derive using implicit differentiation the derivative $d/dx \operatorname{arctanh}(x)$, where

$$\tanh(x) = \sinh(x)/\cosh(x) .$$

The **hyperbolic sine** and **hyperbolic cosine** are defined as are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. We have $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

5 The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y. Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$. Estimate $g(1.01, 0.99)$ using linear approximation.

Lecture 12: Gradient

The **gradient** of a function $f(x, y)$ is defined as

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \ .$$

For functions of three dimensions, define

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle \ .$$

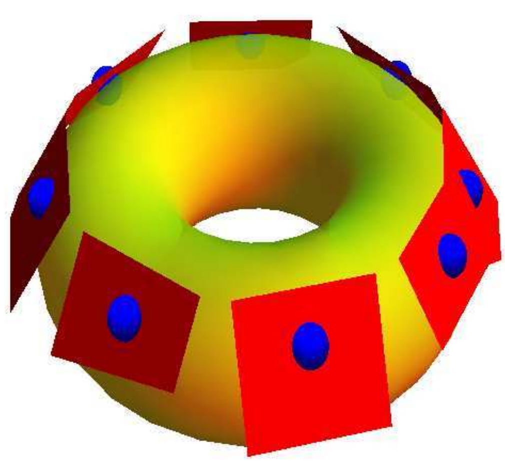
The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. Here is a very important fact:

Gradients are orthogonal to level curves and level surfaces.

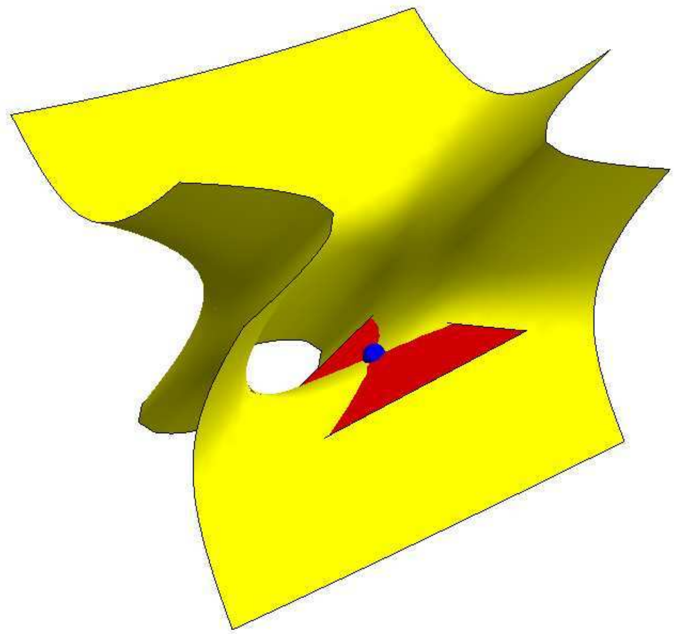
Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$.
Because $\vec{n} = \nabla f(p, q) = \langle a, b \rangle$ is perpendicular to the level curve $f(x, y) = c$ through (p, q), the equation for the tangent line is $ax + by = d$, $a = f_x(p, q)$, $b = f_y(p, q)$, $d = ap + bq$. Compactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus. since it provides a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines:



1 Compute the tangent plane to the surface $3x^2y+z^2-4=0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z\rangle$. And $\nabla f(1,1,1) = \langle 6,3,2\rangle$. The plane is $6x+3y+2z=d$ where d is a constant. We can find the constant d by plugging in a point and get $6x+3y+2z=11$.



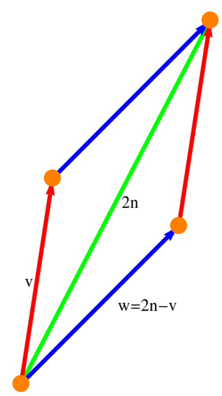
2 **Problem:** reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1\rangle$ at the surface

$$x^4+y^2+z^6=6 .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t=2$ in the point $(-1,-2,1)$. The velocity vector in that ray is $\vec{v} = \langle -1,-1,0\rangle$ The normal vector at this point is $\nabla f(-1,-2,1) = \langle -4,-4,6\rangle = \vec{n}$. The reflected vector is

$$R(\vec{v} = 2\text{Proj}_{\vec{n}}(\vec{v}) - \vec{v} .$$

We have $\text{Proj}_{\vec{n}}(\vec{v}) = 8/68\langle -4,-4,6\rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4,-4,6\rangle - \langle -1,-1,0\rangle$.



If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

The name directional derivative is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt}D_{\vec{v}}f = \frac{d}{dt}f(x + t\vec{v})$.

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}}T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x, y)$, then $D_{\vec{v}}h(x, y)$ gives us the slope we drive on.

3 If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)}f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

4 For $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}}f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

The directional derivative satisfies $\boxed{|D_{\vec{v}}f| \leq |\nabla f||\vec{v}|}$ because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}|$.

The direction $\vec{v} = \nabla f/|\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

If $\vec{v} = \nabla f/|\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

5 You are on a trip in a air-ship over Cambridge at $(1, 2)$ and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?
Solution: The gradient $\nabla p(x, y) = \langle 2x, 4y \rangle$ at the point $(1, 2)$ is $\langle 2, 8 \rangle$. Normalize to get the direction $\langle 1, 4 \rangle/\sqrt{17}$.

The directional derivative has the same properties than any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f + g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + fD_v(g)$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **critical points** and help to understand the function f.

6 The Matterhorn is a 4'478 meter high mountain in Switzerland. It is quite easy to climb with a guide because there are ropes and ladders at difficult places. Evenso there are quite many climbing accidents at the Matterhorn, this does not stop you from trying an

ascent. In suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by the function $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the south-east direction $v = \langle 1, -1\rangle/\sqrt{2}$. Calculate the rate of change in that direction. We have $\nabla f(x, y) = \langle -2x, -2y\rangle$, so that $\langle 20, -20\rangle \cdot \langle 1, -1\rangle/\sqrt{2} = 40/\sqrt{2}$. This is a place, with a ladder, where you climb $40/\sqrt{2}$ meters up when advancing 1m forward.
The rate of change in all directions is zero if and only if $\nabla f(x, y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f/|\nabla f|$ and get $D_{\nabla f} f = |\nabla f|$.

7 Assume we know $D_v f(1, 1) = 3/\sqrt{5}$ and $D_w f(1, 1) = 5/\sqrt{5}$, where $v = \langle 1, 2\rangle/\sqrt{5}$ and $w = \langle 2, 1\rangle/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f.
Solution: Let $\nabla f(1, 1) = \langle a, b\rangle$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.

Homework

1 Find the directional derivative $D_{\vec{v}} f(2, 1) = \nabla f(2, 1) \cdot \vec{v}$ into the direction $\vec{v} = \langle -3, 4\rangle/5$ for the function $f(x, y) = x^5 y + y^3 + x + y$.

2 A surface $x^2 + y^2 - z = 1$ radiates light away. It can be parametrized as $\vec{r}(x, y) = \langle x, y, x^2 + y^2 - 1\rangle$. Find the parametrization of the wave front which is distance 1 from the surface.

3 Assume $f(x, y) = 1 - x^2 + y^2$. Compute the directional derivative $D_{\vec{v}}(x, y)$ at $(0, 0)$, where $\vec{v} = \langle \cos(t), \sin(t)\rangle$ is a unit vector. Now compute

$$D_v D_v f(x, y)$$

at $(0, 0)$, for any unit vector. For which directions is this **second directional derivative** positive?

4 The **Kitchen-Rosenberg formula** gives the curvature of a level curve $f(x, y) = c$ as

$$\kappa = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

Use this formula to find the curvature of the ellipsoid $f(x,y) = x^2 + 2y^2 = 1$ at the point $(1,0)$.
This formula is useful in computer vision. If you want to derive the formula, you can check that the angle

$$g(x,y) = \arctan(f_y/f_x)$$

of the gradient vector has κ as the directional derivative in the direction $\vec{v} = \langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2}$ tangent to the curve.

5 One numerical method to find the maximum of a function of two variables is to move in the direction of the gradient. This is called the **steepest ascent method**. You start at a point (x_0, y_0) then move in the direction of the gradient for some time c to be at $(x_1, y_1) = (x_0, y_0) + c\nabla f(x_0, y_0)$. Now you continue to get to $(x_2, y_2) = (x_1, y_1) + c\nabla f(x_1, y_1)$. This works well in many cases like the function $f(x,y) = 1 - x^2 - y^2$. It can have problems if the function has a flat ridge like in the **Rosenbrock function**

$$f(x,y) = 1 - (1-x)^2 - 100(y - x^2)^2 .$$

Plot using a computer the Contour map of this function on $-0.6 \leq x \leq 1, -0.1 \leq y \leq 1.1$ and find the directional derivative at $(1/5, 0)$ in the direction $(1,1)/\sqrt{2}$. Why is it also called the **banana function**?