8/7/2014 PRACTICE EXAM VI **Maths 21a, O. Knill, Summer 2014**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

1) T F For any two nonzero vectors $\vec{v}, \vec{w}$ the vector $\vec{v} - \vec{w}$ is perpendicular to $\vec{v} \times \vec{w}$.

Solution:
Indeed, both $\vec{v}$ and $\vec{w}$ are perpendicular. So also their difference.

2) T F The cross product satisfies the law $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.

Solution:
Take $\vec{v} = \vec{w}$, then the right hand side is the zero vector while the left hand side is not zero in general (for example if $u = i, v = j$).

3) T F If the curvature of a smooth curve $\vec{r}(t)$ in space is defined and zero for all t, then the curve is part of a line.

Solution:
One can see that with the formula $\kappa(t) = |r'(t) \times r''(t)|/|r'(t)|^3$ which shows that the acceleration $r''(t)$ is in the velocity direction at all times. One can also see it intuitively or with the definition $\kappa(t) = |T'(t)|/|r'(t)|$. If curve is not part of a line, then T has to change which means that κ is not zero somewhere.

4) T F The curve $\vec{r}(t) = (1-t)A + tB, t \in [0,1]$ connects the point A with the point B.

Solution:
The curve is a parameterization of a line and for $t = 0$, one has $\vec{r}(0) = A$ and for $t = 1$ one has $\vec{r}(1) = B$.

5) T F For every c, the function $u(x,t) = (2\cos(ct) + 3\sin(ct))\sin(x)$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$.

Solution:
Just differentiate.

6) T F The arc length of $\vec{r}(t) = (t, \sin(t))$, $t \in [0, 2\pi]$ is $\int_0^{2\pi} \sqrt{1 + \cos^2(t)}\, dt$.

Solution:
The speed at time t is $|\vec{r}'(t)| = \sqrt{1+\cos^2(t)}$.

7) | T | **F** | Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then $f_{xx}(x_0, y_0) < 0$.

Solution:
While this would be true for $g(x,y) = f(y)$, where the constraint is a straight line parallel to the y axes, it is false in general.

8) | T | **F** | The function $f(x,y,z) = x^2 - y^2 - z^2$ decreases in the direction $(2,-2,-2)/\sqrt{12}$ at the point $(1,1,1)$.

Solution:
It **increases** in that direction.

9) | **T** | F | $\vec{F}$ is a vector field for which $|\vec{F}(x,y,z)| \leq 1$. For every curve $C : \vec{r}(t)$ with $t \in [0,1]$, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is $\leq$ the arc length of C.

Solution:
$|\vec{F} \cdot \vec{r}'| \leq |\vec{F}||\vec{r}'| \leq |\vec{r}'|$.

10) | **T** | F | Let $\vec{F}$ be a vector field and C is a curve which is a flow line, then $\int_C \vec{F} \cdot d\vec{r} > 0$.

Solution:
The vector field points in the same direction than the velocity vector so that the dot product is positive at each point.

11) | T | **F** | The divergence of the gradient of any $f(x,y,z)$ is always zero.

Solution:
$\text{div}(\text{grad}(f)) = \Delta f$ is the Laplacian of f.

12) [T] F — For every function f, one has $\text{div}(\text{curl}(\text{grad}(f))) = 0$.

Solution:
Both because $\text{div}(\text{curl}(F) = 0$ and $\text{curl}(\text{grad}(f) = 0$.

13) T [F] — If for two vector fields $\vec{F}$ and $\vec{G}$ one has $\text{curl}(\vec{F}) = \text{curl}(\vec{G})$, then $\vec{F} = \vec{G} + (a, b, c)$, where a, b, c are constants.

Solution:
One can also have $\vec{F} = \vec{G} + \text{grad}(f)$ which are vectorfields with the same curl.

14) T [F] — For every vector field $\vec{F}$ the identity $\text{grad}(\text{div}(\vec{F})) = \vec{0}$ holds.

Solution:
$F = (x^2, y^2, z^2)$ has $\text{div}(F) = (2x, 2y, 2z)$ which has a nonzero gradient.

15) [T] F — If a nonempty quadric surface $g(x, y, z) = ax^2 + by^2 + cz^2 = 5$ can be contained inside a finite box, then $a, b, c \geq 0$.

Solution:
If one or two of the constants a, b, c are negative, we have a hyperboloid which all can not be contained into a finite space. If all three are negative, then the surface is empty.

16) T [F] — If $\vec{F}$ is a vector field in space then the flux of $\vec{F}$ through any closed surface S is 0.

Solution:
While it is true that the flux of $\text{curl}(F)$ vanishes through every closed surface, this is not true for $\vec{F}$ itself. Take for example $F = (x, y, z)$.

17) T [F] — If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z), then $\text{curl}(\vec{F}) = (0, 0, 0)$ for all (x, y, z).

Solution:
Take $(-y, x, 0)$ for example.

18) | T | **F** | The flux of the vector field $\vec{F}(x, y, z) = (y+z, y, -z)$ through the boundary of a solid region E is equal to the volume of E.

Solution:
By the divergence theorem, the flux through the boundary is $\int\int\int_E \text{div}(F)\ dV$ but $\text{div}(F) = 0$. So the flux is zero.

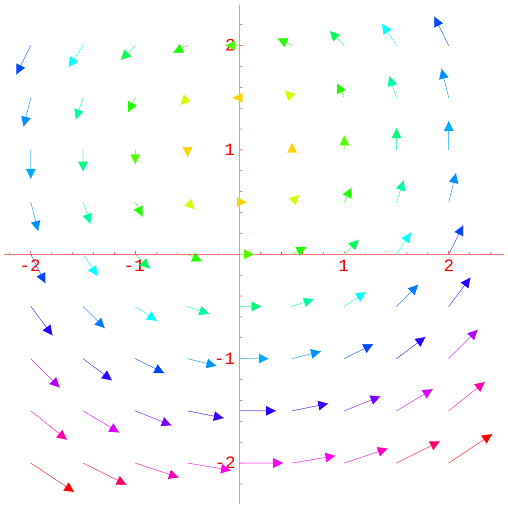
19) | **T** | F | If in spherical coordinates the equation $\phi = \alpha$ (with a constant α) defines a plane, then $\alpha = \pi/2$.

Solution:
Otherwise, it is would be a cone (or for $\alpha = 0$ or $\alpha = \pi$ a half line).

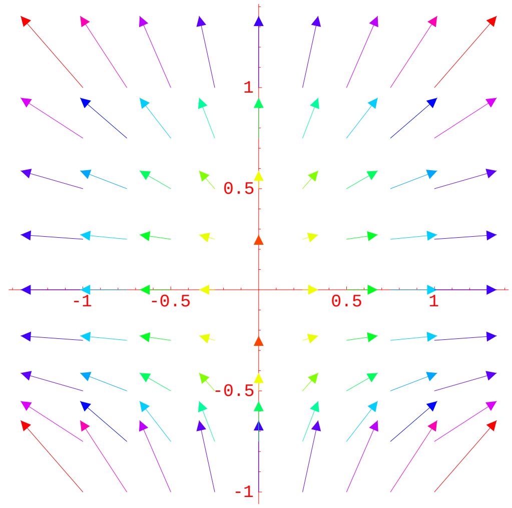
20) | **T** | F | For every function $f(x, y, z)$, there exists a vector field $\vec{F}$ such that $\text{div}(\vec{F}) = f$.

Solution:
In order to solve $P_x + Q_y + R_z = f$ just take $F = (0, 0, \int_0^z f(x, y, w)\ dw)$.

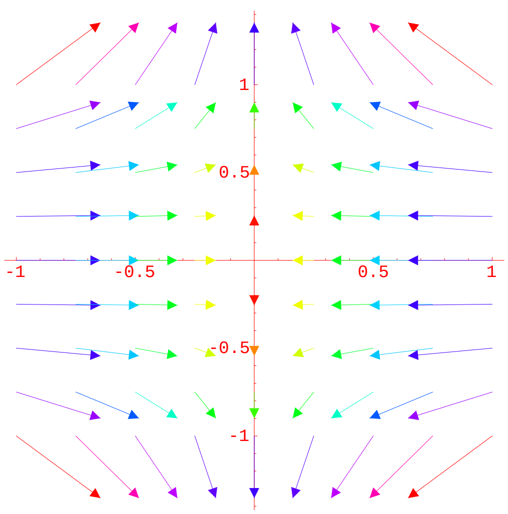
Problem 2) (10 points)



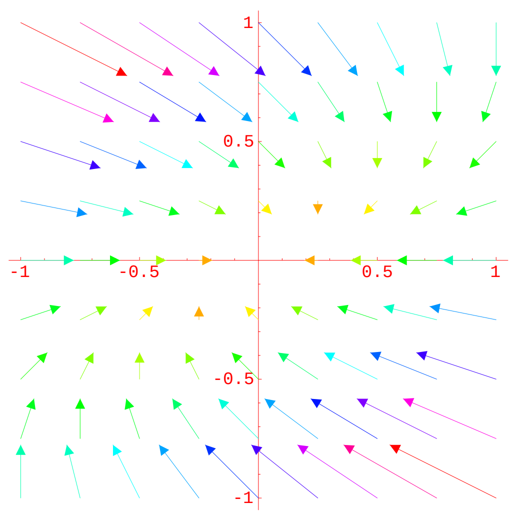
I



II



III



IV

For the sign of the curl or divergence, where either + (positive), $-$ (negative) or 0 for zero. The vector fields are considered on the square $[-1/2, 1/2]x[-1/2, 1/2]$ in this problem.

Enter I,II,III,IV here	Vector field	curl sign	divergence sign
	$F(x,y) = (x, y^2)$		
	$F(x,y) = (1-y, x)$		
	$F(x,y) = (y-x, -y)$		
	$F(x,y) = (-x, y^3)$		

Solution:

Enter I,II,III,IV here	Vector field	curl sign	divergence sign
II	$F(x,y) = (x, y^2)$	0	+
I	$F(x,y) = (1-y, x)$	+	0
IV	$F(x,y) = (y-x, -y)$	-	-
III	$F(x,y) = (-x, y^3)$	0	-

Problem 3) (10 points)

Mark with a cross in the column below "conservative" if a vector fields is conservative (that is if $\text{curl}(\vec{F})(x,y,z) = (0,0,0)$ for all points (x,y,z)). Similarly, mark the fields which are incompressible (that is if $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z)). No justifications are needed.

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x,y,z) = (-5, 5, 3)$		
$\vec{F}(x,y,z) = (x, y, z)$		
$\vec{F}(x,y,z) = (-y, x, z)$		
$\vec{F}(x,y,z) = (x^2+y^2, xyz, x-y+z)$		
$\vec{F}(x,y,z) = (x-2yz, y-2zx, z-2xy)$		

Solution:

Vectorfield	conservative $\text{curl}(\vec{F}) = \vec{0}$	incompressible $\text{div}(\vec{F}) = 0$
$\vec{F}(x, y, z) = (-5, 5, 3)$	X	X
$\vec{F}(x, y, z) = (x, y, z)$	X	
$\vec{F}(x, y, z) = (-y, x, z)$		
$\vec{F}(x, y, z) = (x^2 + y^2, xyz, x - y + z)$		
$\vec{F}(x, y, z) = (x - 2yz, y - 2zx, z - 2xy)$	X	

Problem 4) (10 points)

Let E be a parallelogram in three dimensional space defined by two vectors $\vec{u}$ and $\vec{v}$.

a) (3 points) Express the diagonals of the parallelogram as vectors in terms of $\vec{u}$ and $\vec{v}$.

b) (3 points) What is the relation between the length of the crossproduct of the diagonals and the area of the parallelogram?

c) (4 points) Assume that the diagonals are perpendicular. What is the relation between the lengths of the sides of the parallelogram?

Solution:
a) first diagonal $\boxed{u + v}$, second diagonal $\boxed{u - v}$.
b) $(u + v) \times (u - v) = 2v \times u =$ $\boxed{\text{2 times area of of parallelogram}}$.
c) $(u + v) \cdot (u - v) = |u|^2 - |v|^2 = 0$, so that $\boxed{|u| = |v|}$.

Problem 5) (10 points)

Find the volume of the largest rectangular box with sides parallel to the coordinate planes that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} = 1$.

Solution:
The volume of the box is $8xyz$. The Lagrange equations are

$$\begin{aligned} 8yz &= \lambda x/2 \\ 8xz &= \lambda 2y/9 \\ 8xy &= \lambda 2z/25 \\ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1 = 0 & \end{aligned}$$

We can solve this by solving the first three equations for λ and expressing y, z by x, plugging this into the fourth equation. An other way to solve this is to multiply the first equation with x, the second with y and third with z.
The solution is $\boxed{x = 2/\sqrt{3}, y = \sqrt{3}, z = 5/\sqrt{3}}$. The maximal volume is $8xyz = 80/\sqrt{3}$.

Problem 6) (10 points)

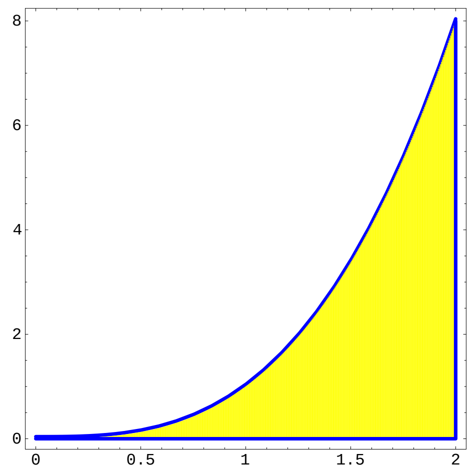
Evaluate

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8} \, dxdy.$$

Solution:
This type II integral can not be computed as it is. We write it as a type I integral: from the boundary relation $x = y^{1/3}$ we obtain $y = x^3$ and $y = 8$ corresponds to $x = 2$:

$$\int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8} \, dydx$$
$$\int_0^2 \frac{x^9}{3} \frac{e^{x^2}}{x^8} \, dx$$
$$\int_0^2 \frac{x}{3} e^{x^2} \, dx$$
$$e^{x^2}/6|_0^2 = (e^4 - 1)/6$$

The result is $\boxed{\frac{e^4-1}{6}}$.



Problem 7) (10 points)

Evaluate $\int\int_D \frac{2xy}{x^2+y^2} \, dxdy$, where D is the intersection of the annulus $1 \leq x^2 + y^2 \leq 2$ with the second quadrant $\{x \leq 0, y \geq 0\}$.

Solution:
Use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ for which $2xy = r^2 \sin(2\theta)$ and $x^2+y^2 = r^2$. The integral is $\int_1^{\sqrt{2}} \int_{\pi/2}^{\pi} r^2 \sin(2\theta)/r^2 \; r d\theta dr = -(r^2)/2|_1^{\sqrt{2}} \cos(2\theta)/2|_{\pi/2}^{\pi} = -1/2$.

Problem 8) (10 points)

a) (3 points) Find all the critical points of the function $f(x, y) = -(x^4 - 8x^2 + y^2 + 1)$.

b) (3 points) Classify the critical points.

c) (2 points) Locate the local and absolute maxima of f.

d) (2 points) Find the equation for the tangent plane to the graph of f at each absolute maximum.

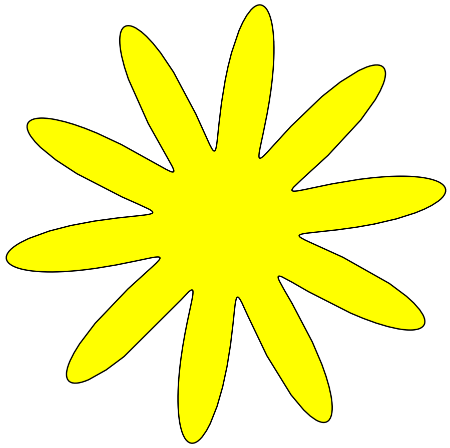
Solution:
a) $\boxed{(\pm 2, 0) \text{ and } (0, 0)}$.
b) $\boxed{(-2, 0) \text{ is a local maximum with value } 15}$.
$\boxed{(0, 0) \text{ is a saddle with value } -1}$.
$\boxed{(2, 0) \text{ is a maximum with value } 15}$.
c) $\boxed{\text{The local maxima are } (\pm 2, 0)}$. They are also the absolute maxima because f decays at infinity.
d) To calculate the tangent plane at the maximum, write the graph of f as a level surface $g(x, y, z) = z - f(x, y)$. The gradient of g is orthogonal to the surface. We have $\nabla g = (0, 0, 1)$ so that the tangent plane has the equation $z = d = const$. Plugging in the point $(\pm 2, 0, 15)$ shows that $\boxed{z = 15}$ is the equation for the tangent plane for both maxima.

Problem 9) (10 points)

Find the area $\int\int_R 1\ dxdy$ of the 10 legged "sea star" R, enclosed by the polar curve

$$r(\theta) = 2 + \sin(10\ \theta) ,$$

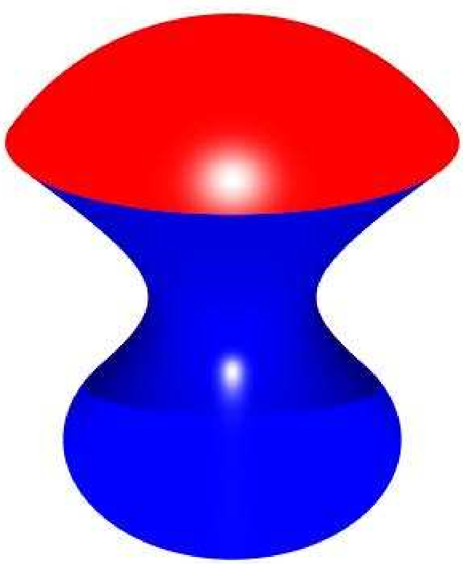
where $\theta \in [0, 2\pi]$. The photo to the right shows a real sea star.

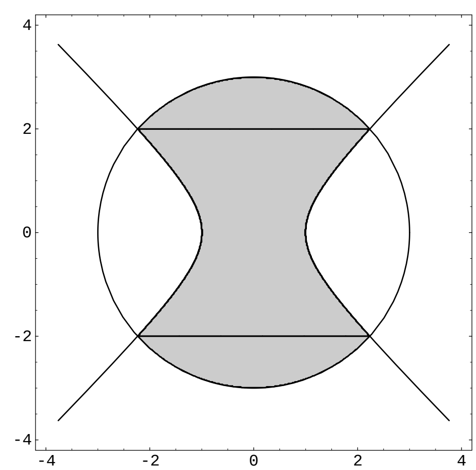


Solution:
$\int_0^{2\pi} \int_0^{2+\sin(10\theta)} r \; dr d\theta = \int_0^{2\pi} (\sin^2(10\theta) + 4 + 4\sin(10\theta))/2 \; d\theta$ which is equal to $(\pi + 8\pi)/2 =$ $\boxed{= 9\pi/2}$.

Problem 10) (10 points)

Find the volume of the intersection of the interior of the one sided hyperboloid $x^2+y^2-z^2 \leq 1$ with the solid ball enclosed by the sphere $x^2 + y^2 + z^2 \leq 9$.





Solution:
The simplest solution is to use cylindrical coordinates and compute the complement of the object with respect to the sphere. Then

$$V = 4\pi 3^3/3 - \int_{-2}^{2} \int_0^{2\pi} \int_{\sqrt{1+z^2}}^{\sqrt{9-z^2}} r \; dr d\theta dz$$

This is $36\pi - 2\pi \int_{-2}^{2} [(9-z^2) - (1+z^2)]/2 \; dz = 36\pi - \pi \int_{-2}^{2} (10 - 2z^2) \; dz = \boxed{44\pi/3}$. An other possibility is to the middle part $-2 \leq z \leq 2$ of the solid seperately and add to this the volumes of the upper and lower caps which have equal volume. It is possible to use spherical coordinates. But cylindrical coordinates are again simpler:

$$V = (2\pi) \int_{-2}^{2} (1+z^2)/2 \; drdz + 2[(2\pi) \int_2^3 (9-z^2)/2 \; drdz] = (2\pi)14/3 + 2[(2\pi)4/3] = \frac{44\pi}{3} .$$

Again, the answer is $\boxed{44\pi/3}$.

Problem 11) (10 points)

Let the curve C be parametrized by $\vec{r}(t) = (t, \sin t, t^2 \cos t)$ for $0 \leq t \leq \pi$. Let $f(x,y,z) = z^2 e^{x+2y} + x^2$ and $\vec{F} = \nabla f$. Find $\int_C \vec{F} \cdot d\vec{r}$.

Solution:
Use the fundamental theorem of line integrals. The result is $f(r(\pi)) - f(r(0)) = f(\pi, 0, -\pi^2) - f(0,0,0) = \pi^4 e^\pi + \pi^2 - 0 = \boxed{\pi^4 e^\pi + \pi^2}$.

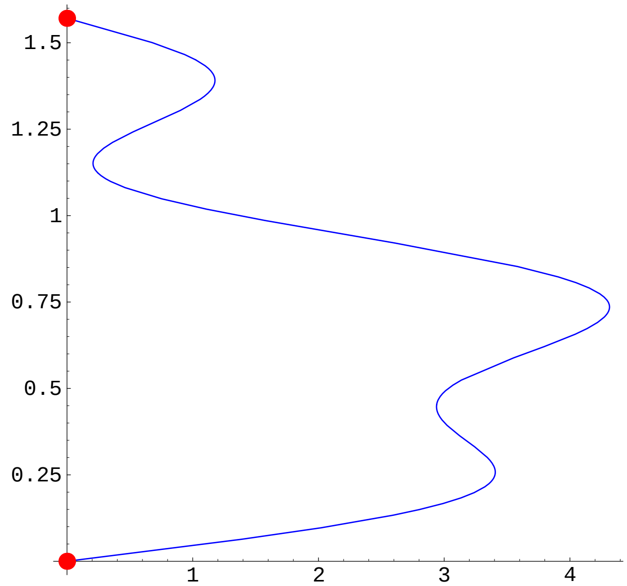
Problem 12) (10 points)

a) Find the linear approximation $L(x,y)$ of $f(x,y) = \sqrt{4 + 2x^2 + 4y^2}$ at the point $(x,y) = (2,1)$.
b) Find the equation for the tangent line to the level curve of $f(u,v)$ at $(2,1)$.

Solution:
a) $L(x,y) = 4 + (x-2) + (y-1) = 1 + x + y$.
b) $1 + x + y = 4$ or $\boxed{x + y = 3}$

Problem 13) (10 points)

Find the line integral of the vector field $\vec{F}(x,y) = \langle x^{30} + y, y^{50} + x \rangle$ along the path $\vec{r}(t) = \langle 4\sin(\pi \sin(t)) + \sin(10t), t \rangle$ with $0 \le t \le \pi/2$.



Solution:
There are two possibilities to solve this problem. The first solution is to find the potential $f(x,y)$ which has F as a gradient field. It is $f(x,y) = \frac{x^{31}}{31} + \frac{y^{51}}{51} + xy$.
By the fundamental theorem of line integrals, the result is $f(0,\pi/2) - f(0,0) = \boxed{\frac{\pi^{51}}{2^{51} \cdot 51}}$.
The second solution is to note that since the vector field has vanishing curl, it is conservative and the line integral along the path C is the same as the line integral along the path $\vec{r}(t) = (0,t)$. We obtain the same answer by computing this line integral.

Problem 14) (10 points)

Evaluate the line integral of the vector field $\vec{F}(x,y) = (y^2, x^2)$ in the clockwise direction around the triangle in the xy-plane defined by the points $(0,0)$, $(1,0)$ and $(1,1)$ in two ways:

a) (5 points) by evaluating the three line integrals.
b) (5 points) using Greens theorem.

Solution:
The problem asks to do this in the clockwise direction. We do it in the counterclockwise direction and change then the sign.
a) $\int_0^1 F(t,0)\cdot(1,0)\ dt + \int_0^1 F(1,t)\cdot(0,1)\ dt + \int_0^1 F(1-t,1-t)\cdot(-1,-1)\ dt = 0+1-2/3 = 1/3$.
So, the result for the clockwise direction is $\boxed{-1/3}$.
b) The curl of F is $2x-2y$.

$$\int_0^1 \int_0^x (2x-2y) dy\ dx = \int_0^1 2x^2 - x^2\ dx = 1/3$$

So, the result for the clockwise direction is $\boxed{-1/3}$.

Problem 15) (10 points)

Use Stokes theorem to evaluate the line integral of $\vec{F}(x,y,z) = (-y^3, x^3, -z^3)$ along the curve $\vec{r}(t) = (\cos(t), \sin(t), 1-\cos(t)-\sin(t))$ with $t \in [0, 2\pi]$.

Solution:
The curve is contained in the graph of the function $f(x,y) = 1-x-y$ which is parameterized by $r(u,v) = (u,v,1-u-v)$ and has the normal vector $r_u \times r_v = (1,0,-1)\times(0,1,-1) = (1,1,1)$. The curl of F is $(0,0,3x^2+3y^2)$ so that $F(r(u,v))\cdot(r_u\times r_v) = 3(u^2+v^2)$. The surface is parameterized over the region $R = \{u^2+v^2 \leq 1\}$ and $\int\int_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^{2\pi} 3r^3\ d\theta dr = \boxed{\frac{3\pi}{2}}$.